"In the beginning of the year 1665 I found the method of approximating series and the rule for reducing any dignity of any binomial to such a series. The same year in May I found the method of tangents of Gregory and Slusius, and in November had the direct method of Fluxions, and the next year in January had the Theory of Colours, and in May following I had entrance into the inverse method of Fluxions, and in the same year I began to think of gravity extending to the orb of the moon...and having thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the earth, and found them to answer pretty nearly. All this was in the two years of 1665 and 1666, for in those years I was in the prime of my age for invention and minded Mathematicks and Philosophy more than at any time since."

Sir Isaac Newton (1642 – 1727)

8 Gravitation and Central Forces

While Newtonian mechanics can in theory describe motion under any force, an analytic solution to a problem is often difficult to come by. Until now, we have generally focused on constant forces such as gravity near Earth's surface, which are trivial to integrate. In this chapter, we will explore a more general class of forces called *central forces*. The most important central force is the gravitational force between astronomical bodies, which was proposed by Isaac Newton as a model of planetary motion. Additionally, we will show that a gravitational force between two bodies leads to orbits that are described by *Kepler's laws*. We will then discuss how bodies interacting under a central force can scatter at various angles, just like the collisions we investigated in Chapter 5. Finally, we will discuss how the two body central force problem can be reduced to an effective one body problem.

8.1 Properties of Central Forces

Gravitation is a common example of a special class of forces known as *central forces*. A central force points radially inward or outward from some source point, and is only dependent on the distance from that source. We can represent a central force as $\boldsymbol{F} = F(r)\hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is the radial unit vector in polar coordinates.

One important property of central forces is that they conserve angular momentum about the source point. We can see this by calculating the torque due to a central force, which is given by $\boldsymbol{\tau} = r\hat{\boldsymbol{r}} \times F(r)\hat{\boldsymbol{r}} = 0$.

Another feature of central forces is that they are conservative. We can prove this by showing that the curl is zero. An arbitrary central force can be written as

$$\boldsymbol{F} = F(r)\hat{\boldsymbol{r}}.$$

Using the expression for the curl in spherical coordinates given in Table A.1, we have

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\boldsymbol{\theta}} - \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\boldsymbol{\phi}} = 0.$$

Therefore, we can write every central force as $\boldsymbol{F} = -\boldsymbol{\nabla} U$ for some potential energy U.

8.2 Gravity as a Central Force

So far, we have modeled gravity as a constant force pointing towards the earth's surface. However, in reality, gravitation is more complicated – a force of gravitation exists between any two bodies and always attracts them. Newton determined that gravitation is a central force given by

$$\boldsymbol{F}_{1,2} = -\frac{GM_1M_2}{r_{1,2}^2}\hat{\boldsymbol{r}},$$

where M_1 and M_2 are the masses of the two bodies and $r_{1,2}$ is the distance between them, as indicated in Figure 8.1. The constant G has been experimentally determined to have a value of $6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$.

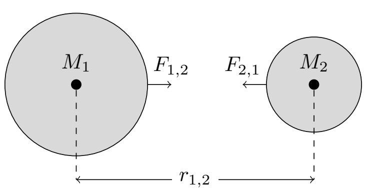


Figure 8.1: Gravity is an attractive force between two masses that is inversely proportional to the square of their separation.

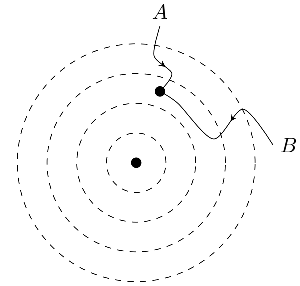


Figure 8.2: The equipotentials of the field of a central force are concentric spheres.

The negative sign in the force law is due to the direction of the $\hat{\boldsymbol{r}}$ vector, which points from the body causing the gravity towards the body experiencing it. Moreover, we note that this force law is consistent with Newton's third law: $\boldsymbol{F}_{2,1} = -\boldsymbol{F}_{1,2}$.

Since gravitation is a central force, we can find an associated potential energy function. Physically, we define this potential function as the work done by the gravitational force when an object is brought from an infinite distance to a distance r from the gravitational source. As depicted in Figure 8.2, regardless of what path we choose, we get the same value for the potential. In this case, values of constant potential lie on spheres centered around the source. We call these *equipotential surfaces*. It is easy to see that equipotential surfaces for any central force are concentric spheres.

The value of the potential at a distance r from the source is given by

$$U = -\int_C \boldsymbol{F}(r)\cdot d\boldsymbol{s} = -\int_C F(r)\hat{\boldsymbol{r}}\cdot\left(dr\hat{\boldsymbol{r}} + r\,d\phi\hat{\boldsymbol{\phi}} + r\sin\phi\,d\theta\hat{\boldsymbol{\theta}}\right).$$

The spherical unit vectors are orthogonal, so we can immediately evaluate the dot product and obtain

$$U = -\int_\infty^r F(r)\,dr.$$

Substituting the inverse square central force law for gravitation, we find

$$U = -\int_\infty^r -\frac{GM_1M_2}{r^2}\,dr = -\frac{GM_1M_2}{r}.$$

Regardless of the value of r, the potential energy function for gravity is negative. This leads us to an important concept. If an object is subject to a planet's gravitational pull, then it takes work to free the object from the planet. In terms of energy conservation, we can think of this as the inability to reach a net positive energy due to the large negative potential energy of gravitational attraction. Thus, there is a minimum velocity, known as the *escape velocity*, at which an object will be able to escape the influence of gravity and move away indefinitely. To determine the escape velocity, we can employ the conservation of energy by equating the sum of the total kinetic and potential energies at two different radial displacements. We will use m as the mass of the small body, and M as the mass of the stationary larger body. This gives

$$\frac{1}{2}mv^2 - \frac{GmM}{r} = \frac{1}{2}mv_f^2 - \frac{GmM}{r_f}.$$

If the object is to reach an infinite distance, then it must have at least as much energy as there is potential energy at infinity. Since we have used infinity as the reference point, the potential energy there is zero, so

$$\frac{1}{2}mv^2 - \frac{GmM}{r} \geq 0.$$

This gives $v \geq \sqrt{\frac{2GM}{r}}$. We call $v_e = \sqrt{\frac{2GM}{r}}$ the escape velocity. Any faster launch speed will also allow the object to escape a body of mass M, but with nonzero kinetic energy at infinity. Physically, this means that the object's velocity will approach a nonzero value.

Example 8.1. Calculate the energy and period of a satellite of mass m which is in a circular orbit around a planet of mass M, as in Figure 8.3.

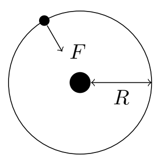


Figure 8.3: A satellite is in a circular orbit of radius R around a planet.

Solution: In order for the object to remain in a circular orbit, the gravity must account for the centripetal acceleration. So,

$$\frac{GMm}{R^2} = \frac{mv^2}{R}.$$

Therefore, the kinetic energy of the body is

$$\frac{1}{2}mv^2 = \frac{GMm}{2R}.$$

Since the gravitational potential energy is given by $-\frac{GMm}{R}$, the total energy is $-\frac{GMm}{2R}$.

To calculate the period, we can substitute $v = \frac{2\pi R}{T}$. This gives

$$\frac{GMm}{R^2} = \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2,$$

so

$$T^2 = \frac{4\pi^2 R^3}{GM}.$$

Example 8.2. A small body with mass m circularly orbits a massive planet with mass M and radius R. The body loses energy at a rate

$$P = -k\frac{(Mm)^2(M+m)}{r^5}.$$

Assuming the mass starts at a radius r_0 and the orbit remains nearly circular, determine the time t_* it takes for the body to collide with a planet.

Solution: Using the expression we have derived for the energy of a body in a circular orbit,

$$\frac{dE}{dt} = -\frac{GMm}{2}\frac{d}{dt}\frac{1}{r} = -k\frac{(Mm)^2(M+m)}{r^5}.$$

This simplifies to the differential equation

$$r^3\frac{dr}{dt} = -\frac{2Mm(M+m)k}{G}.$$

Separating and integrating, we find

$$\frac{1}{4}\left(R^4 - r_0^4\right) = -\frac{2Mm(M+m)kt_*}{G},$$

so

$$t_* = \frac{G}{8Mm(M+m)k}\left(r_0^4 - R^4\right).$$

Einstein's theory of general relativity predicts an energy loss at this rate; but, except in extreme cases, k is so small that the effect is undetectable.

8.3 Gravity of Large Bodies

Our expression for gravitational force, $\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$, does not take the spatial extent of a body into account. When considering the attraction of an object to Earth, we might expect that the mass directly underneath the object matters a great deal, while the mass on the other side of Earth does not attract it much at all. However, for a spherically symmetric body, there is an important result that allows us to ignore the extent of the attracting body and treat it as a point mass. We will derive this result, and then show how to account for slight deviations from spherical symmetry.

The Shell Theorem

In order to find the gravitational force of a spherical body on a point mass m, we can simply integrate the field due to each differential mass of the spherical body, as shown in Figure 8.4.

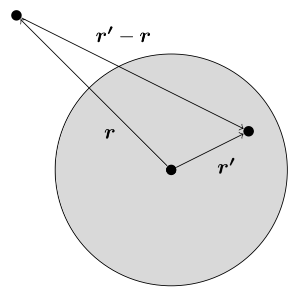


Figure 8.4: The gravitational field of a spherical body is the integral of the fields due to each differential mass element.

This gives

$$\boldsymbol{F}(\boldsymbol{r}) = -\int \frac{Gm\rho(\boldsymbol{r'})\,(\boldsymbol{r}-\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|^3}\,d\boldsymbol{r'},$$

where ρ is the mass density. This is a very cumbersome integral. While we certainly could define a spherical coordinate system and evaluate it using the law of cosines, it is much easier to use some of the results of vector calculus (covered in Appendix A.4).

We will start by defining the gravitational field of a particle. If the gravitational field of a particle is $\boldsymbol{g}(\boldsymbol{r})$, then the force it exerts on a mass m at a location $\boldsymbol{r}$ is

$$\boldsymbol{F} = m\boldsymbol{g}(\boldsymbol{r}).$$

From this, it is easy to see that the field for an object of mass M is given by

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM\hat{\boldsymbol{r}}}{r^2}.$$

We will now use the divergence theorem to determine the value of this field. The divergence of $\boldsymbol{g}(\boldsymbol{r})$ can be computed using the expression in Table A.1:

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = \frac{1}{r^2}\frac{\partial(r^2 g_r)}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}(-GM) = 0.$$

However, this analysis does not hold at the origin, where there is a singularity in the field. Clearly, we should expect a negative divergence at the origin, since all the field lines point inwards. We can determine a volume integral of this divergence using the divergence theorem, which states that for an arbitrary vector field $\boldsymbol{F}$,

$$\iiint \boldsymbol{\nabla}\cdot\boldsymbol{F}\,dV = \oiint \boldsymbol{F}\cdot d\boldsymbol{S}.$$

Choosing our surface to be a sphere centered at the origin, the surface integral becomes trivial since $\boldsymbol{g}\cdot d\boldsymbol{S}$ is constant:

$$\iiint \boldsymbol{\nabla}\cdot\boldsymbol{g}\,dV = \left(4\pi r^2\right)\left(-\frac{GM}{r^2}\right) = -4\pi GM.$$

We can now use this result (sometimes known as *Gauss's Law for gravitation*) to analyze a continuous mass distribution. In this case, a point has a mass dM, so we can write

$$\iiint \boldsymbol{\nabla}\cdot\boldsymbol{g}\,dV = -4\pi G\,dM.$$

This result holds no matter how small the sphere around the mass is. Thus, we can contract the sphere to a differential volume dV, and we are left with a result known as *Poisson's equation*,

$$\boldsymbol{\nabla}\cdot\boldsymbol{g}(\boldsymbol{r}) = -4\pi G\frac{dM}{dV} = -4\pi G\rho(\boldsymbol{r}).$$

Now, consider an object with density given by $\rho(\boldsymbol{r}) = f(r)$; that is, the object is spherically symmetric, so the density depends only upon the radius. We can then apply the divergence theorem again, this time to determine the surface integral of the gravitational field. We will start by taking a spherical surface with radius greater than the radius of the body. Then, we have

$$\oiint \boldsymbol{g}\cdot d\boldsymbol{S} = \iiint \boldsymbol{\nabla}\cdot\boldsymbol{g}\,dV = -4\pi G\iiint \rho\,dV = -4\pi GM.$$

A very similar process with a sphere of radius less than that of the body gives

$$\oiint \boldsymbol{g}\cdot d\boldsymbol{S} = -4\pi GM_{\text{enc}},$$

where M_{enc} is the total mass of the portion of the object contained within this sphere. Since the object is spherically symmetric, the gravitational field will depend only on r. Additionally, we know that it will be directed inwards. So, $\boldsymbol{g}\cdot d\boldsymbol{S} = -g\,dA$, and we have

$$-4\pi r^2 g = -4\pi GM_{\text{enc}}.$$

Therefore, the magnitude of $\boldsymbol{g}$ must be

$$g(r) = \frac{GM_{\text{enc}}}{r^2}.$$

This is also true when r exceeds the radius of the body, as long as we understand M_{enc} to be the total mass M. Since we know the direction of $\boldsymbol{g}$, we can finally write

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM_{\text{enc}}\hat{\boldsymbol{r}}}{r^2}.$$

This is called the *shell theorem*. For a spherically symmetric object, the gravitational field at a radius r is the same as the field due to a point mass at the center of the object, with its mass equal to the total mass enclosed within a spherical shell of radius r. This allows us to easily analyze gravitational fields of astronomical bodies, which are most often spherically symmetric.

Example 8.3. A hole of negligible radius is drilled through the center of Earth. If an object of mass m is dropped down one side of the hole, determine how long it takes for it to reach the other end. Assume that there is no friction, and that Earth has radius R, mass M, and a uniform density.

Solution: The shell theorem will immediately give us the gravitational field at any position in the hole. The total mass enclosed at radius r is

$$M_{\text{enc}} = \left(\frac{r}{R}\right)^3 M,$$

so we have

$$g = -\frac{GM}{R^3}r.$$

Therefore, the force on the mass is

$$F = -\frac{GMm}{R^3}r.$$

This has the same form as a spring force, with $k = \frac{GMm}{R^3}$. We thus expect simple harmonic oscillation with a period $T = 2\pi\sqrt{\frac{m}{k}}$. The time it takes for the object to reach the other side is a half period, or

$$T = \pi\sqrt{\frac{R^3}{GM}} = 42\,\text{min}.$$

This phenomenon, called a "gravity train," was first proposed by Robert Hooke in the 17th century. Since then, the idea has been rejected because of the immense engineering challenges involved and the friction inside the earth.

Quadrupole Moment Correction

All astronomical bodies of considerable size are approximately spherical. However, for some precise calculations, the deviations from a perfect sphere are important.

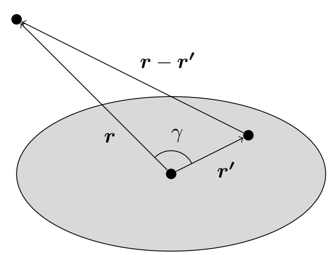


Figure 8.5: The gravitational field of a non-spherical body can be found by integrating.

In the non-spherical case, it is generally easier to determine the gravitational potential per mass, which we denote $u(\boldsymbol{r})$, than the gravitational field. Referring to Figure 8.5, the required integral to calculate the gravitational potential is

$$u(\boldsymbol{r}) = -\int \frac{G\rho(\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|}\,dV' = -G\int \frac{\rho(\boldsymbol{r'})}{\sqrt{r^2 - 2rr'\cos\gamma + r'^2}}\,dV',$$

where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r'}$. Defining $x = \frac{r'}{r}$, this becomes

$$u(\boldsymbol{r}) = -\frac{G}{r}\int \frac{\rho(\boldsymbol{r'})}{\sqrt{1 - 2x\cos\gamma + x^2}}\,dV'.$$

We can expand the integrand as a power series in x. If $x < 1$, which corresponds to determining the potential outside the body, then we can use the binomial expansion

$$(1 + (x^2 - 2x\cos\gamma))^{-\frac{1}{2}} = 1 - \frac{1}{2}\left(x^2 - 2x\cos\gamma\right) + \frac{3}{8}\left(x^2 - 2x\cos\gamma\right)^2 - \ldots.$$

Collecting powers of x, this becomes[i]

$$(1 + (x^2 - 2x\cos\gamma))^{-\frac{1}{2}} = 1 + (\cos\gamma)x + \frac{1}{2}\left(3\cos^2\gamma - 1\right)x^2 + \ldots.$$

Now, substituting into the integral, we have

$$\begin{aligned} u(\boldsymbol{r}) = &-\frac{GM}{r} - \frac{G}{r^2}\int \rho(\boldsymbol{r'})r'\cos\gamma\,dV' \\ &-\frac{G}{2r^3}\int \rho(\boldsymbol{r'})r'^2\left(3\cos^2\gamma - 1\right)\,d\boldsymbol{r'} + \ldots. \end{aligned}$$

The second term can be made to vanish by choosing the center of mass of the body as the origin. All the other terms contribute corrections to the zero-order approximation $-\frac{GM}{r}$. In this section, we will only concern ourselves with the second-order correction:

$$\Delta u = -\frac{G}{2r^3}\int \rho(\boldsymbol{r'})r'^2\left(3\cos^2\gamma - 1\right)\,dV'.$$

This is called the *gravitational quadrupole* term.

The quadrupole term is usually written in terms of a quadrupole moment matrix, which has components

$$Q_{ij} = \int \rho(\boldsymbol{r})\left(3r_ir_j - r^2\delta_{ij}\right)\,dV,$$

where δ_{ij} is the Kronecker delta symbol,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}.$$

You will verify in Problem 13 that the quadrupole term is equal to

$$-\frac{G}{2r^3}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij}n_in_j,$$

where $n_i = \frac{r_i}{r}$.

[i] The complete expansion is $\sum_{\ell=0}^{\infty} P_\ell(\cos\gamma)x^\ell$, where P_ℓ is a Legendre polynomial of degree ℓ (see Appendix C, page 532).

Example 8.4. The most prominent deviation from a sphere for most planets is the equatorial bulge. This occurs when the rotation of a planet leads to the equatorial radius being larger than the polar radius. Consider a planet of uniform density with mass M, equatorial radius R_e, and polar radius R_p. Determine the quadrupole correction to the gravitational potential.

Solution: To compute the quadrupole term, we need to find each component of the quadrupole matrix. Integrating over an ellipsoid is difficult, so we will start by mapping the planet to a unit sphere through the coordinate transformations

$$x' = \frac{x}{R_e}, \quad y' = \frac{y}{R_e}, \quad z' = \frac{z}{R_p}.$$

Then, for example, we have

$$\begin{aligned}Q_{11} &= \int \rho(\boldsymbol{r})\left(3x^2 - r^2\right) dV \\ &= \frac{3M}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \Big(R_e^2 \sin^2\theta(2\cos^2\phi - \sin^2\phi) \\ &\quad - R_p^2\cos^2\theta\Big) r^4 \sin\theta\, dr\, d\theta\, d\phi.\end{aligned}$$

Integrating, we have

$$\begin{aligned}Q_{11} &= \frac{3M}{20\pi}\int_0^{2\pi}\int_0^{\pi} \Big(R_e^2 \sin^3\theta(2\cos^2\phi - \sin^2\phi) \\ &\quad - R_p^2\cos^2\theta\sin\theta\Big)\, d\theta\, d\phi \\ &= \frac{3M}{20\pi}\int_0^{2\pi}\left(\frac{4}{3}R_e^2(2\cos^2\phi - \sin^2\phi) - \frac{2}{3}R_p^2\right) d\phi \\ &= \frac{M}{5}\left(R_e^2 - R_p^2\right).\end{aligned}$$

Similarly, you can show that the other diagonal elements are

$$\begin{aligned}Q_{22} &= \frac{M}{5}\left(R_e^2 - R_p^2\right), \\ Q_{33} &= \frac{2M}{5}\left(R_p^2 - R_e^2\right).\end{aligned}$$

It is simple to show that, due to the symmetry of the body, the off-diagonal elements all vanish. So, the quadrupole correction is

$$\frac{G}{2r^5}\frac{M}{5}\left(R_p^2 - R_e^2\right)\left(x^2 + y^2 - 2z^2\right).$$

Example 8.5. A hoop in the xy plane with mass M has a radius a. A smaller mass m circularly orbits the hoop a distance $R \gg a$ away.

(a) Determine the orbital frequency of the mass to order $\left(\frac{a}{R}\right)^2$ if the orbital plane of the mass coincides with the plane of the hoop.

(b) Assume the hoop is spun with a very large angular velocity about the x axis. Determine the orbital frequency of the mass to order $\left(\frac{a}{R}\right)^2$ if the mass orbits in the yz plane.

Solution: First, we will compute the quadrupole correction to the field of the hoop. We call the plane of the hoop $z = 0$, so the matrix elements involving z are zero. Then, the Q_{11} element is

$$\begin{aligned}Q_{11} &= \int \rho(\boldsymbol{r})\left(3x^2 - r^2\right) dV \\ &= \frac{M}{2\pi a}\int_0^{2\pi} a^3(3\cos^2\theta - 1)\, d\theta \\ &= \frac{Ma^2}{2}.\end{aligned}$$

Similarly, $Q_{22} = \frac{Ma^2}{2}$. It is easy to show that the off-diagonal elements are zero, so the potential per mass is

$$u = -\frac{GM}{r} - \frac{G}{2r^3}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij}n_i n_j = -\frac{GM}{r} - \frac{GMa^2}{4r^5}(x^2 + y^2).$$

In the plane of the hoop, we have $z = 0$ and $r^2 = x^2 + y^2$, so

$$F = -m\frac{\partial u}{\partial r} = -\frac{GMm}{r^2} - \frac{3GMma^2}{4r^4}.$$

Setting this equal to the centripetal acceleration, we have

$$m\Omega^2 R = \frac{GMm}{R^2} + \frac{3GMma^2}{4R^4},$$

or

$$\Omega \approx \Omega_0\left(1 + \frac{3a^2}{8R^2}\right).$$

When the hoop is rapidly spinning, the potential energy per mass at any given time is the same as before, except with $x^2 + y^2$ replaced with the distance to the center of the hoop in the plane of the hoop. When the hoop is rotated by an angle θ, its plane is normal to $\hat{\boldsymbol{n}} = \sin\theta\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\boldsymbol{k}}$, so

$$\begin{aligned}u &= -\frac{GM}{r} - \frac{GMa^2}{4r^5}\left(r^2 - (\boldsymbol{r}\cdot\hat{\boldsymbol{n}})^2\right) \\ &= -\frac{GM}{r} - \frac{GMa^2}{4r^5}\left(x^2 + y^2\cos^2\theta + z^2\sin^2\theta - 2yz\sin\theta\cos\theta\right).\end{aligned}$$

The angle θ oscillates with a frequency $\omega \gg \Omega$, so the mass is influenced by the average value of u rather than its instantaneous value. Using $\langle\sin^2\theta\rangle = \langle\cos^2\theta\rangle = \frac{1}{2}$ and $\langle\sin\theta\cos\theta\rangle = 0$, we have

$$\langle u\rangle = -\frac{GM}{r} - \frac{GMa^2}{4r^5}\left(x^2 + \frac{y^2 + z^2}{2}\right).$$

Since the mass orbits in the yz plane, this simplifies to

$$\langle u \rangle = -\frac{GM}{r} - \frac{GMa^2}{8r^3}.$$

Note that spinning the hoop was necessary to make the potential rotationally symmetric in the yz plane, allowing the mass to maintain a circular orbit. Solving for Ω in the same way as before, we find

$$\Omega \approx \Omega_0 \left(1 + \frac{3a^2}{16R^2}\right).$$

Example 8.6. A satellite circularly orbits Earth at an angular velocity $\Omega^2 = \frac{GM}{r^3}$. The satellite can be modeled as a thin cylinder with a radius a and a length $2L \gg a$. The length of the satellite is much smaller than the distance from the satellite to Earth.

(a) Determine the equilibrium position of the satellite in a frame rotating with the satellite.

(b) Determine the frequency of small oscillations if the satellite is perturbed in the plane of its orbit.

(c) Determine the frequency of small oscillations if the satellite is perturbed perpendicular to the plane of its orbit.

(d) Determine the equilibrium position of the satellite if the satellite is spun about its long axis with a small angular velocity ω.

Solution: To make this problem simpler, we will briefly explain a method which we develop much more fully in Chapter 12. We know that an object orbiting with frequency Ω in a circle of radius r experiences a centripetal acceleration $-\omega^2 \boldsymbol{r}$. If we work in the frame of the orbiting object, the relative acceleration vanishes because the particle is at rest. Mathematically, this occurs because the frame also has an acceleration $-\omega^2 \boldsymbol{r}$, so the difference of the accelerations is zero. From this we derive an important relationship: starting from Newton's second law in the rest frame, $\boldsymbol{F} = m\boldsymbol{a}$, we can substitute $\boldsymbol{a} = -\omega^2 \boldsymbol{r} + \boldsymbol{a}_r$, where $\boldsymbol{a}_r$ is the relative acceleration. Rearranging, we have $\boldsymbol{F} + m\omega^2 \boldsymbol{r} = m\boldsymbol{a}_r$. This means that if we work in the rest frame, we must add an outward centrifugal force $m\omega^2 \boldsymbol{r}$, or equivalently a potential energy $-\frac{1}{2} m\omega^2 r^2$.

By symmetry, the only possible possible equilibrium positions of the satellite are the horizontal or vertical positions. In the rotating frame, the gravitational attraction as well as the repulsion due to the centripetal force cause a net torque. Because gravitational attraction increases and centripetal repulsion decreases closer to the earth, perturbations from the vertical position would experience a restoring torque, while perturbations from the horizontal configuration would experience a net destabilizing torque. Therefore, the vertical position is the equilibrium position.

When perturbed slightly from the vertical position in the direction of the orbital plane, as shown in Figure 8.6, the gravitational potential of the satellite is

$$U_g = -GM \int_{-L}^{L} \frac{\lambda\, ds}{\sqrt{r^2 + s^2 + 2rs\cos\theta}},$$

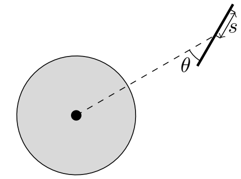


Figure 8.6: The satellite is perturbed by an angle θ from its vertical equilibrium.

where $\lambda = \frac{m}{2L}$ is the linear density of the satellite, θ is the angular deviation from the vertical position, and s is the distance to the center of mass of the satellite. As in the derivation of the quadrupole correction, we can expand the integrand as a power series in $\frac{s}{r}$. This gives

$$U_g \approx -\frac{GMm}{r} - \frac{GM}{2r^3} \int_{-L}^{L} \lambda s^2 (3\cos^2\theta - 1)\, ds,$$

or

$$U_g \approx -\frac{GMm}{r} - \frac{GMmL^2}{6r^3}(3\cos^2\theta - 1).$$

In the rotating frame, we also have to account for the potential energy due to the centripetal force, as explained above. This is proportional to the distance squared, so

$$U_c = \frac{\Omega^2}{2} \int_{-L}^{L} \lambda \left(r^2 + s^2 + 2rs\cos\theta\right)\, ds,$$

or

$$U_c = \frac{m\Omega^2 r^2}{2} + \frac{m\Omega^2 L^2}{6}.$$

The net torque is given by

$$\tau = -\frac{\partial}{\partial\theta}(U_g + U_c) = -\frac{GMmL^2}{r^3} \sin\theta\cos\theta.$$

The moment of inertia of the satellite about its center of mass is $\frac{1}{3} mL^2$, so

$$\frac{1}{3} mL^2 \ddot{\theta} = -\frac{GMmL^2}{r^3} \sin\theta\cos\theta \approx -\frac{GMmL^2}{r^3}\theta,$$

which corresponds to a frequency of oscillation $\omega_1 = \Omega\sqrt{3}$.

Perpendicular to the orbital plane, we have the same expression for the gravitational potential energy. However, the potential energy due to the centripetal force is now

$$U_c = \frac{\Omega^2}{2} \int_{-L}^{L} \lambda \left(r + s\cos\theta\right)^2\, ds,$$

or

$$U_c = \frac{m\Omega^2 r^2}{2} + \frac{m\Omega^2 L^2}{6} \cos^2\theta.$$

The net torque is then

$$\tau = -\frac{\partial}{\partial\theta}(U_g + U_c) = -\frac{2}{3}mL^2\Omega^2 \sin\theta\cos\theta.$$

This corresponds to a frequency of oscillations $\omega_2 = \Omega\sqrt{2}$.

Now the satellite is given a small angular velocity ω. We will work in the original frame, so we do not need to worry about the torque due to the centripetal force. The rotational kinetic energy of the satellite is given by

$$K_R = \frac{1}{2}I_3\left(\omega + \Omega\sin\theta\right)^2 + \frac{1}{2}I\Omega^2\cos^2\theta,$$

where we have decomposed the orbital angular velocity of the satellite along its principal axes. Using $I_3 = \frac{1}{2}ma^2$ and $I = \frac{1}{3}mL^2$, we have

$$K_R = \frac{1}{4}ma^2\left(\omega + \Omega\sin\theta\right)^2 + \frac{1}{6}mL^2\Omega^2\cos^2\theta.$$

The total energy consists of this term as well as the quadrupole potential term and the kinetic energy of the satellite's center of mass. The latter correction is independent of θ, so setting $\frac{\partial E}{\partial\theta} = 0$ gives

$$\frac{1}{2}ma^2\left(\omega + \Omega\sin\theta\right)\Omega\cos\theta - \frac{1}{3}mL^2\Omega^2\cos\theta\sin\theta + mL^2\Omega^2\cos\theta\sin\theta = 0.$$

Using $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, we have

$$|\theta| \approx \frac{3\omega}{4\Omega}\frac{a^2}{L^2},$$

where we have neglected higher order corrections. Clearly, the orientation of the satellite is perpendicular to its orbit, and $|\theta|$ is small given our initial assumptions.

8.4 Kepler's Laws

Astronomers were analyzing planetary orbits long before Newton's time. One of these astronomers was Johannes Kepler. Using empirical data gathered by Tycho Brahe, Kepler determined three features of planetary orbits, known now as *Kepler's laws*:

1. The orbit of a planet is a conic section, with the Sun located at one of the foci.
2. The segment connecting the sun and the planet sweeps out equal areas during equal time intervals.
3. The square of the orbital period of a planet is proportional to the cube of the semimajor axis of its orbit.

Luckily, we do not have to repeat Kepler's painstaking data analysis. Using Newton's laws and what we have understood about central forces, we can prove each of Kepler's statements.

The first statement may seem simplest, but it is the most challenging to prove. We will start by considering a planet of mass m orbiting a star of mass M under the influence of gravity. The total energy associated with this motion (assuming the star is so massive that it stays approximately fixed) is given by

$$E = \frac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v} + U(r),$$

where $U(r) = -\frac{C}{r}$ is the potential energy associated with the gravitational force[ii], and $v = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$. Performing the dot product gives

$$E = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right) + U(r).$$

The second term in the kinetic energy is the energy associated with the rotational component of the planet's motion, so we call it the azimuthal kinetic energy. Recalling that angular momentum is conserved under the action of a central force, we know that $L = mr^2\frac{d\theta}{dt}$ is a constant. We can rewrite the second term as $\frac{1}{2}m\left(r\frac{d\theta}{dt}\right)^2 = \frac{L^2}{2mr^2}$. This gives

$$\begin{aligned} E &= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} + U(r) \\ &= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + U_e(r), \end{aligned}$$

where we have defined the effective potential

$$U_e(r) \equiv \frac{L^2}{2mr^2} - \frac{C}{r}.$$

This eliminates the explicit dependence on θ, so we can rearrange and write

$$\sqrt{\frac{2}{m}\left(E - U_e(r)\right)} = \frac{dr}{dt}.$$

This is a separable differential equation and can easily be reduced to an integral. However, carrying out the integration is very difficult. In fact, most central forces do not have clean analytic solutions for their equations of motion. While numerical integration is a viable option, it does not give nearly as much information as an analytic result.

We can avoid evaluating the time integral by focusing on determining the shape of the orbit. This requires us to solve for the radius as a function of the angle rather than the time. To do this, we can use the chain rule to rewrite our differential equation:

$$\sqrt{\frac{2}{m}\left(E - U_e(r)\right)} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{L}{mr^2}.$$

[ii] In this section we will let $C = GMm$ in order to simplify notation.

After squaring both sides, we obtain

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2m(E - U_e(r))}{L^2}.$$

We can substitute the effective potential for gravity, $U_e(r) = \frac{L^2}{2mr^2} - \frac{C}{r}$. The equation becomes

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} + \frac{2mC}{rL^2} - \frac{1}{r^2}.$$

We will now perform a crucial change of variables by letting $u = \frac{1}{r}$. The right hand side is easy to rewrite in terms of u, but $\frac{dr}{d\theta}$ is less obvious. For this, we need to use the chain rule:

$$\frac{dr}{d\theta} = \frac{dr}{du}\frac{du}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}.$$

Therefore, we obtain

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2mE}{L^2} + \frac{2mC}{L^2}u - u^2.$$

Another change of variables is in order: we can let $v = u - \frac{mC}{L^2}$. The equation becomes

$$\begin{aligned}\left(\frac{dv}{d\theta}\right)^2 &= \frac{2mE}{L^2} + \frac{2mC}{L^2}\left(v + \frac{mC}{L^2}\right) - \left(v + \frac{mC}{L^2}\right)^2 \\ &= \frac{2mE}{L^2} - v^2 + \frac{m^2C^2}{L^4} \\ &= -v^2 + \frac{m^2C^2}{L^4}\left(1 + \frac{2EL^2}{mC^2}\right).\end{aligned}$$

Before solving this equation, we define $k = \frac{mC}{L^2}\sqrt{1 + \frac{2EL}{mC^2}}$. So, we have

$$\frac{dv}{d\theta} = \sqrt{k^2 - v^2}.$$

Separating and integrating, and letting the initial angle be θ_0, we have

$$\int \frac{dv}{\sqrt{k^2 - v^2}} = \int d\theta,$$

so that

$$v = k\cos(\theta - \theta_0).$$

Substituting $v = \frac{1}{r} - \frac{mC}{L^2}$, we have

$$r = \frac{L^2}{mC}\frac{1}{1 + e\cos(\theta - \theta_0)},$$

where we have defined the eccentricity e to be $\sqrt{1 + \frac{2EL^2}{mC^2}}$. In Appendix D, it is shown that this is the general polar form of a conic section, so we have proved Kepler's first law. The shape of the orbit is dictated by the value of e, so we will examine each possibility.

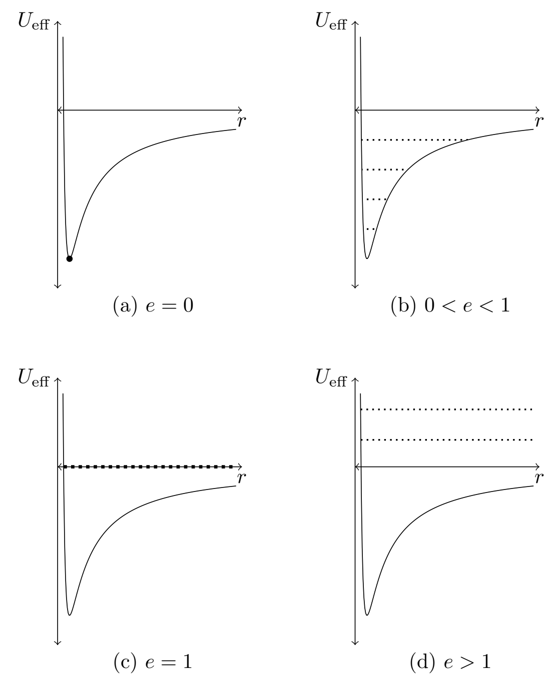


Figure 8.7: The effective potential graph describes the different kinds of orbits.

- $e = 0$: In this case we see that r is constant, which implies that the orbit is circular. Using the expression for eccentricity, we also see that $E = -\frac{mC^2}{2L^2}$. Since this is negative, the gravitational potential energy must be larger than the planet's kinetic energy. If we plot the effective potential as a function of radius as in Figure 8.7, zero eccentricity corresponds to a trajectory that remains at the bottom of the well. Thus, there is no radial kinetic energy, and the azimuthal kinetic energy and potential energy remain constant, as we would expect.

- $0 < e < 1$: This corresponds to elliptical orbits. For the orbit to form an ellipse, we see that the energy must satisfy $-\frac{mC^2}{2L^2} < E < 0$. On the effective potential diagram, this corresponds to energy lines in the well. Since the kinetic energy is always positive, the effective potential energy must lie below the energy line. Thus, the planet will oscillate between two distinct radii, which confirms that the orbit is an ellipse. The minimum and maximum radii clearly occur at $\theta - \theta_0 = 0$ and $\theta - \theta_0 = \pi$, and are

$$r_{\min} = \frac{L^2}{mC(1+e)},$$
$$r_{\max} = \frac{L^2}{mC(1-e)}.$$

- $e = 1$: This corresponds to a parabolic orbit. Physically, this implies that the total energy of the system must be zero. This also means that the kinetic energy of the system is zero at $r = \infty$ since the potential energy vanishes at infinity. Thus, the object passes the origin at a minimum distance $r_{\min} = \frac{L^2}{2mC}$, swings around, and escapes to infinity with its velocity approaching zero.

- $e > 1$: This corresponds to a hyperbolic orbit. Because $E > 0$, the particle will have energy left over even as it reaches infinity. The object will swing around the origin at a distance $r < \frac{L^2}{2mC}$, and its trajectory will have a straight asymptote.

Since planetary orbits are bound, their paths must be elliptical (or circular).

Example 8.7. Determine possible orbits for the central force $F(r) = -\frac{\beta}{2r^3}$.

Solution: The potential associated with this central force is $U(r) = -\frac{\beta}{r^2}$. The effective potential is then given by

$$U_e(r) = \frac{L^2}{2mr^2} - \frac{\beta}{r^2}.$$

The is equivalent to the effective potential for gravity, with $C = 0$ and $\frac{L^2}{2m} \to \frac{L^2}{2m} - \beta$. Therefore,

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{L^2 - 2m\beta}{L^2}\frac{1}{r^2}.$$

Calling $u = \frac{1}{r}$, we have

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2mE}{L^2} - \left(1 - \frac{2m\beta}{L^2}\right)u^2.$$

If we call $k^2 = \frac{2mE}{L^2 - 2m\beta}$, then

$$\int \frac{du}{\sqrt{k^2 - u^2}} = \sqrt{1 - \frac{2m\beta}{L^2}} \int d\theta.$$

The solutions fall into three possible categories. When $E > 0$ and $L^2 > 2m\beta$, the orbits will behave qualitatively like a hyperbola, reaching a minimum distance from the source before escaping to infinity. In this case, the orbit is

$$\frac{1}{r} = \sqrt{\frac{2mE}{L^2 - 2m\beta}} \cos\left((\theta - \theta_0)\sqrt{1 - \frac{2m\beta}{L^2}}\right).$$

You can verify that the minimum distance to the source is $\sqrt{\frac{L^2 - 2m\beta}{2mE}}$. When $E > 0$ and $L^2 < 2m\beta$, the mass will start at infinity and spiral arbitrarily close to the source. This orbit is given by

$$\frac{1}{r} = \sqrt{\frac{2mE}{2m\beta - L^2}} \sinh\left((\theta - \theta_0)\sqrt{\frac{2m\beta}{L^2} - 1}\right).$$

Finally, when $E < 0$ and $L^2 < 2m\beta$, the mass will start a finite distance from the source before spiraling arbitrarily close to it. This orbit is given by

$$\frac{1}{r} = \sqrt{\frac{2mE}{L^2 - 2m\beta}} \cosh\left((\theta - \theta_0)\sqrt{\frac{2m\beta}{L^2} - 1}\right).$$

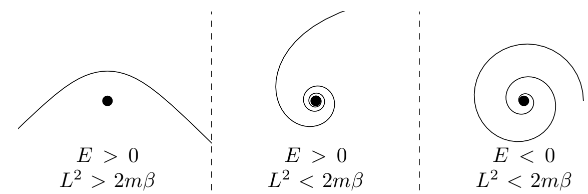


Figure 8.8: The three possible orbits for a force $F(r) \propto r^{-3}$.

Kepler's second law, the law of equal areas, is a consequence of conservation of angular momentum. Figure 8.9 shows a small slice of a planetary orbit. Its area is given by a cross product of the position vector and the displacement:

$$dA = \frac{1}{2}|\boldsymbol{r} \times d\boldsymbol{r}|.$$

Thus, the time derivative of the area is

$$\frac{dA}{dt} = \frac{1}{2}\left|\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt}\right| = \frac{L}{2m}.$$

Because the total angular momentum of the orbiting body remains constant, the rate at which area is swept out is constant. This proves Kepler's second law.

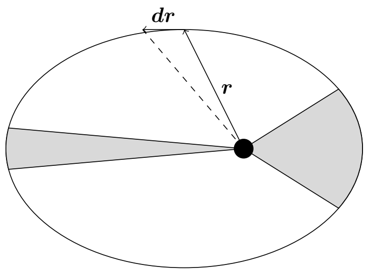


Figure 8.9: Equal areas are swept out in equal times because $\boldsymbol{L}$ is conserved.

Example 8.8. Consider the central force attraction

$$\ddot{w} = -k|w|^a \frac{w}{|w|},$$

where w is taken to be a complex number, with its magnitude representing r and its phase representing θ. Using the transformation $z = w^\beta$ and imposing Kepler's second law, show that this central force equation can be mapped to

$$\ddot{z} = -\kappa|z|^b \frac{z}{|z|},$$

and find the relationship between a, b, and β.

Solution: Since both orbits conserve angular momentum, Kepler's second law must hold, so

$$\frac{\frac{dA}{dt}}{\frac{dA'}{d\tau}} = \frac{\frac{1}{2}|w|^2 \frac{d\theta}{dt}}{\frac{1}{2}|w|^{2\beta} \frac{d\theta'}{d\tau}} = C,$$

where C is a constant. Because $z = w^\beta$, $\theta' = \beta\theta$, and thus

$$|w|^{2-2\beta} \frac{d\tau}{dt} = C\beta,$$

If we call $C\beta = \alpha$, then $\frac{d\tau}{dt} = \alpha|w|^{2\beta-2}$. Therefore,

$$\begin{aligned}
\frac{d^2 z}{d\tau^2} &= \frac{1}{\alpha^2}|w|^{2-2\beta} \frac{d}{dt}\left(|w|^{2-2\beta} \frac{dw^\beta}{dt}\right), \\
&= \frac{\beta}{\alpha^2}|w|^{2-2\beta} \frac{d}{dt}\left(\bar{w}^{1-\beta} \frac{dw}{dt}\right), \\
&= \frac{\beta}{\alpha^2}|w|^{2-2\beta} \left((1-\beta)\bar{w}^{-\beta}\left|\frac{dw}{dt}\right|^2 + \bar{w}^{1-\beta} \frac{d^2 w}{dt^2}\right).
\end{aligned}$$

Factoring out $2(1-\beta)\bar{w}^{-\beta}$, and using $\frac{d^2 w}{dt^2} = -k\frac{w}{|w|^{1-a}}$, we have

$$\frac{d^2 z}{d\tau^2} = \frac{2\beta(1-\beta)}{\alpha^2} \bar{w}^{-\beta}|w|^{2-2\beta} \left(\frac{1}{2}\left|\frac{dw}{dt}\right|^2 - \frac{k}{2(1-\beta)} \frac{1}{|w|^{-1-a}}\right).$$

The quantity in the parentheses is very similar to the energy, which we can write as

$$E = \frac{1}{2}\left|\frac{dw}{dt}\right|^2 + \frac{k}{1+a} \frac{1}{|w|^{-1-a}}.$$

These quantities are equal when $a + 1 = 2\beta - 2$, or $\beta = \frac{a+3}{2}$. If this is the case, then we can use the conservation of energy to obtain

$$\frac{d^2 z}{d\tau^2} = -\kappa\bar{w}^{-\beta}|w|^{2-2\beta} = -\kappa\frac{w^\beta}{|w|^{4\beta-2}},$$

where κ is a constant expressible in terms of β and E. Because $z = w^\beta$,

$$\frac{d^2 z}{d\tau^2} = -\kappa\frac{z}{|z|^{4-\frac{2}{\beta}}} = -\kappa|z|^{\frac{2}{\beta}-3} \frac{z}{|z|}.$$

Therefore, $b = \frac{2}{\beta} - 3$, or $(a+3)(b+3) = 4$. This is an example of *duality*, the mathematical equivalence of two physical systems up to an analytic transformation. Substituting the exponent for the gravitational potential, $a = -2$, we find $b = 1$, which corresponds to a system obeying Hooke's law. Therefore, we would expect to be able to analytically solve a central force obeying Hooke's law by transforming Kepler orbits. Indeed, you can verify that Hooke's law leads to closed elliptical orbits, just like Newton's law of gravitation.

We can use Kepler's second law to prove his third law. Because $\frac{dA}{dt} = \frac{L}{2m}$, $A = \frac{LT}{2m}$, where T is the period of the orbit. Since A represents the total area traversed after one period, it is simply equal to the area of the ellipse, πab. We can conveniently write b in terms of a as $b = a\sqrt{1-e^2}$ (see Appendix D), so that we have

$$\frac{LT}{2m} = \pi a^2\sqrt{1-e^2}.$$

Rearranging slightly, we obtain

$$\pi^2 a^4 = \frac{T^2}{4m} \frac{L^2}{m(1-e^2)}.$$

We can simplify this further, by relating the semimajor axis and the angular momentum. Since the semimajor axis is simply the average of the minimum and maximum radii given earlier,

$$a = \frac{1}{2}(r_{\min} + r_{\max}) = \frac{L^2}{mC(1-e^2)}.$$

Therefore, our equation becomes

$$\pi^2 a^4 = \frac{T^2}{4m} aC,$$

or

$$T^2 = \frac{4\pi^2 m a^3}{C}.$$

This shows that $T^2 \propto a^3$, which is Kepler's third law. It is interesting that we have a formula for the period which is independent of the semiminor axis length. This is reasonable because the semimajor axis, the semiminor axis, and the eccentricity are related by $1 + \left(\frac{b}{a}\right)^2 = e^2$, and the eccentricity is determined by physical parameters such as the angular momentum and energy. Thus, we only need a single length parameter to describe the orbit and its period. Note also that for gravity $C = GMm$, the proportionality constant does not depend on the mass of the orbiting body. This allowed Kepler to observe the relationship in the planets of the solar system, which have widely varying values of m.

We can use this result to derive an important relationship between the energy and the length of the semimajor axis. If we take the equation $\frac{LT}{2m} = \pi a^2 \sqrt{1-e^2}$ and substitute our expression for the eccentricity, we obtain

$$\pi^2 a^4 = -\frac{T^2 C^2}{8Em}.$$

Substituting for T^2 with Kepler's third law, we obtain

$$\pi^2 a^4 = -\left(\frac{C^2}{8Em}\right)\frac{4\pi^2 m a^3}{C},$$

or

$$E = -\frac{C}{2a} = -\frac{GMm}{2a}.$$

Note that these results agree with Example 8.1, since the semimajor axis of a circle is its radius. If we think of this in another way, we have a surprising result: when computing the energy and the period, we can ignore the shape of an orbit, and simply treat an ellipse as a circle with radius a.

Example 8.9. A rope orbits Earth and is aligned in the radial direction. The distance from the center of Earth and the nearest end of the rope is R and the length of the rope is L. Where is the rope most likely to break?

Solution: Let the angular velocity of the rope be Ω. If we consider a small portion of the rope, Newton's second law gives us

$$T(r) - T(r + dr) + \frac{GM}{(R+r)^2}\lambda\, dr = \Omega^2 (R+r)\lambda\, dr,$$

where λ is the linear density of the rope, and M is the mass of Earth. This gives the differential equation

$$\frac{dT}{dr} = \frac{GM}{(R+r)^2}\lambda - \Omega^2(R+r)\lambda.$$

At $r = 0$, the tension in the rope is zero because the front end of the rope is free. Thus,

$$T(r) = GM\lambda\left(\frac{1}{R} - \frac{1}{R+r}\right) - \frac{\Omega^2\lambda}{2}\left((R+r)^2 - R^2\right).$$

By the same logic, the tension must be zero at $r = L$. This allows us to solve for Ω,

$$\Omega^2 = \frac{2GM}{R(R+L)(2R+L)}.$$

The point at which the rope is most likely to break is the point of greatest tension. Setting $\frac{dT}{dr} = 0$ gives

$$(R+r)^3 = \frac{GM}{\Omega^2} = \frac{R(R+L)(2R+L)}{2}.$$

Therefore,

$$r = R\left(\sqrt[3]{(1+\eta)\left(1+\frac{\eta}{2}\right)} - 1\right),$$

where $\eta = \frac{L}{R}$.

Example 8.10. An object moves in a circular orbit which passes through the center of the force C, as shown in Figure 8.10. Give an expression for the central force acting. Assume that the particle's mass is m, the radius of its orbit is R, and its angular momentum is L.

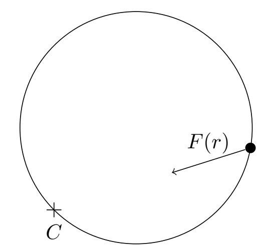


Figure 8.10: A body moves in a circular orbit due to a central force from an object at C.

Solution: The angular momentum about C is conserved. Using the geometry shown in Figure 8.11, we can determine the angular momentum at a given point to be

$$L = mvr\sin\frac{\theta}{2}.$$

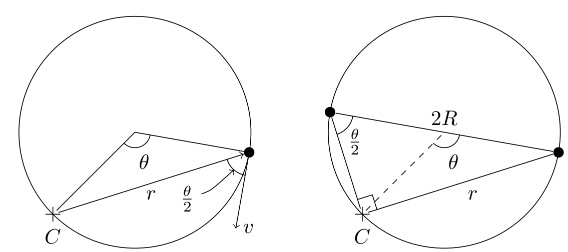


Figure 8.11: The angle between $\boldsymbol{r}$ and $\boldsymbol{v}$ is $\frac{\theta}{2}$ (left), and $\csc\frac{\theta}{2} = \frac{2R}{r}$ (right).

Moreover, energy is conserved:

$$E = \frac{1}{2}mv^2 + U(r) = \frac{L^2}{2mr^2}\csc^2\frac{\theta}{2} + U(r).$$

Figure 8.11 shows that $\csc\frac{\theta}{2} = \frac{2R}{r}$, so we have

$$E = \frac{2L^2R^2}{mr^4} + U(r).$$

We can find the force by taking the derivative with respect to r on both sides, noting that E is constant:

$$F(r) = -\frac{\partial U}{\partial r} = \frac{\partial}{\partial r}\left(\frac{2L^2R^2}{mr^4}\right) = -\frac{8L^2R^2}{mr^5}.$$

Thus, this strange orbit results from a force proportional to $\frac{1}{r^5}$.

Example 8.11. Two point masses m are separated by a distance ℓ and have initial velocities as indicated in Figure 8.12. The masses interact only through universal gravitation. Determine the condition for the masses to collide, and the time it takes for them to do so.

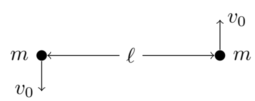


Figure 8.12: Two point masses with a separation ℓ and opposite velocities v_0 collide after some time t.

Solution: In order for two point masses to collide, the distance between them must become zero. For typical effective potential curves, like the ones in Figure 8.7, this is impossible: the effective potential is infinite at $r = 0$. This is because, when r is small, the azimuthal kinetic energy term (which depends on r^{-2}) dominates over the potential energy term (which depends on r^{-1}). The only situation in which this does not occur is when the angular momentum is zero, so that the azimuthal kinetic energy $\frac{L^2}{2mr^2}$ vanishes. In this case, the effective potential becomes $-\infty$ at zero, and so there is no energy barrier preventing a collision. This immediately gives the condition $v_\theta = v_0 = 0$, or that the masses will not rotate.

To determine the time it takes for the two bodies to collide, we can use the conservation of energy. By the conservation of momentum, the two bodies always have equal and opposite velocities. So,

$$-\frac{Gm^2}{\ell} = -\frac{Gm^2}{\ell'} + 2\left(\frac{1}{2}mv^2\right)$$

or

$$Gm\left(\frac{1}{\ell'} - \frac{1}{\ell}\right) = v^2.$$

The rate at which the distance between the objects decreases is $2v$. Thus, $v = -\frac{1}{2}\frac{d\ell'}{dt}$, and we have the differential equation

$$-2\sqrt{Gm\left(\frac{1}{\ell'} - \frac{1}{\ell}\right)} = \frac{d\ell'}{dt}.$$

Separating and integrating, the time is

$$t = -\int_\ell^0 \frac{d\ell'}{2\sqrt{Gm\left(\frac{1}{\ell'} - \frac{1}{\ell}\right)}}.$$

This can be evaluated using special functions. The result is

$$t = \frac{\pi}{4}\frac{\ell^{3/2}}{\sqrt{Gm}}.$$

The collision time is proportional to $\ell^{3/2}$, which is reminiscent of Kepler's third law.

8.5 Central Force Scattering

Kepler was concerned with planetary orbits, which are bounded. However, in proving his laws, we have also determined the form of unbounded orbits. When a mass approaches a gravitational source and $E > 0$, it will be deflected rather than trapped. The angle ϕ by which it is deflected is called the *scattering angle*.

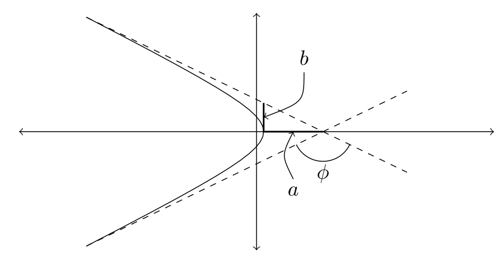


Figure 8.13: An inverse square central force scatters incoming particles along hyperbolic paths.

We will determine the scattering angle by examining a hyperbolic orbit. Figure 8.13 shows the hyperbolic orbit of a mass around a body at the origin. The scattering angle is shown as the angle between the initial and final velocity vectors. It is easy to see geometrically that its value is

$$\phi = \pi - 2\tan^{-1}\left(\frac{b}{a}\right).$$

We can write this in terms of the physical parameters of the system. From the polar form of a

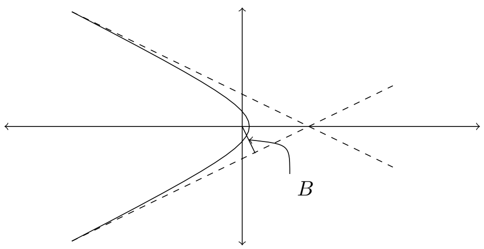


Figure 8.14: The impact parameter B is the distance of the straight line path from the force center.

hyperbola, we know that as r approaches infinity, $\cos\theta$ approaches $-\frac{1}{e}$. Since $\frac{b}{a}$ represents the slope of the asymptote, we have

$$\frac{b}{a} = \tan\theta_\infty = \sqrt{\frac{1}{\cos^2\theta_\infty} - 1} = \sqrt{e^2 - 1}.$$

Recalling our expression for the eccentricity, this becomes

$$\frac{b}{a} = \sqrt{\frac{2EL^2}{mC^2}} = \frac{v_0 L}{C}.$$

We can simplify this further by determining the value of the angular momentum. When the particle is far from the scatterer, it moves in an approximately straight line, so its angular momentum is

$$L = mv_0 B,$$

where B is the distance of closest approach if the particle were to continue in a straight line. This distance is called the *impact parameter*, and is shown in Figure 8.14. We can determine the value of the impact parameter through a geometric argument. If we rotate the right triangle with legs a and b in Figure 8.13 counter-clockwise by $\frac{\pi-\phi}{2}$, the side of length a is coincident with the dashed line. Since $\sqrt{a^2+b^2} = c$, the vertex of the triangle is rotated to the origin. Therefore, the remaining leg of the triangle is $B = b$. The scattering angle is thus

$$\phi = \pi - 2\tan^{-1}\left(\frac{v_0(mv_0 b)}{GMm}\right) = \pi - 2\tan^{-1}\left(\frac{v_0^2 b}{GM}\right).$$

Note that this is independent of the mass of the scattered particle, as we would expect. Additionally, you can verify that the expression gives sensible results for extreme cases. For example, when b or v_0 is very large, the scattering angle is reduced almost to zero.

In practice, we usually cannot exactly control the impact parameter. Instead, we generally have to deal with a beam of particles with a large range of impact parameters. In this situation, it is often more useful to use the *differential cross section* of the scattering. This is defined to be the ratio of the initial cross-sectional area of a beam and the solid angle it subtends after scattering[iii], as shown in Figure 8.15.

[iii]The solid angle of an area A on a sphere of radius R is $\frac{A}{R^2}$, and is measured in steradians, a two-dimensional analogue of radians. So, for example, the solid angle of a sphere is 4π.

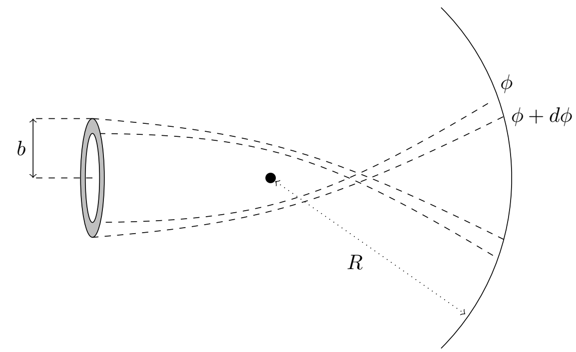


Figure 8.15: The ratio of the area of the ring to the solid angle subtended on the sphere gives the differential cross section.

Consider a ring of particles with impact parameter between b and $b + db$. The ring has a cross-sectional area of $d\sigma = 2\pi b|db|$. To compute its differential cross section, we need to find the solid angle subtended by the scattered beam. We know that

$$\phi = \pi - 2\tan^{-1}\left(\frac{v_0^2 b}{GM}\right),$$

so

$$b = \frac{GM}{v_0^2}\cot\frac{\phi}{2}.$$

Therefore,

$$\left|\frac{db}{d\phi}\right| = \frac{GM}{2v_0^2\sin^2\frac{\phi}{2}}.$$

We now have enough information to determine the solid angle. On a large sphere centered at the scattering mass, the beam becomes a ring of radius $R\sin\phi$ and thickness $R|d\phi|$. So, the solid angle is

$$d\Omega = \frac{2\pi(R\sin\phi)(R|d\phi|)}{R^2} = 2\pi\sin\phi|d\phi|.$$

So, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b|db|}{2\pi\sin\phi|d\phi|} = \frac{b}{\sin\phi}\left|\frac{db}{d\phi}\right| = \frac{GMb}{2v_0^2\sin\phi\sin^2\frac{\phi}{2}}.$$

By substituting for b, we obtain finally

$$\frac{d\sigma}{d\Omega} = \frac{G^2M^2}{4v_0^4\sin^4\frac{\phi}{2}}.$$

You can verify that this also gives a correct qualitative description of scattering. When $\phi \approx \pi$, the cross section is minimized, which implies that only a narrow range of impact parameters lead to beams which scatter directly backwards. On the contrary, when $\phi \approx 0$, the cross section grows without bound. This is because any beam far enough from the scattering mass will barely scatter at all, leading to a diverging value of the derivative.

In addition to the differential result, it is also sometimes useful to find total cross section over which a particle is likely to get scattered. This is accomplished by integrating the differential cross section over the solid angle,

$$\sigma_t = \int \frac{d\sigma}{d\Omega}\, d\Omega = \int \frac{G^2M^2}{4v_0^4 \sin^4 \frac{\phi}{2}}\, d\Omega.$$

Because $d\Omega = 2\pi \sin\phi |d\phi|$, the integral is

$$\sigma_t = \int_0^{\Phi_{\max}} \frac{2\pi G^2 M^2 \sin\phi}{4v_0^4 \sin^4 \frac{\phi}{2}}\, d\phi,$$

where $\Phi_{\max}$ is the maximum possible scattering angle. In the case of an inverse square potential, with $\Phi_{\max} = \pi$, this integral diverges. Thus, a particle will get scattered at any value of the impact parameter or energy. For this reason, we call gravity a long-range force.

Example 8.12. Scattering is a common phenomenon in nuclear physics, since most forces between particles are central. A common approximation for a nuclear scattering force is the spherical well potential, defined by

$$V(r) = \begin{cases} 0 & r > a \\ -V_0 & r \le a \end{cases}.$$

Determine the differential cross section for the scattering of an object under this potential, assuming that the total energy of the object is E_0.

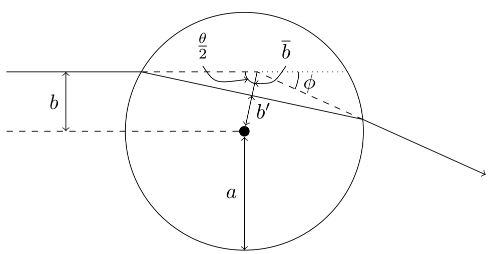


Figure 8.16: A particle scatters through a spherical well potential.

Solution: The trajectory is shown in Figure 8.16. Since the potential is 0 outside a sphere of radius a, the object travels in a straight line. Inside the sphere, the object also travels in a straight line, but with a different slope. Once it emerges out of the sphere, the object will be deflected once more and then continue along a different straight line. In order to determine the deflection of the object inside the sphere, we utilize the conservation of energy

and angular momentum. Conservation of energy implies that the velocity inside the sphere is

$$v = \sqrt{\frac{2}{m}\left(E_0 + V_0\right)}.$$

Conservation of angular momentum about the origin of the sphere tells us that

$$mb\sqrt{\frac{2E_0}{m}} = mb'\sqrt{\frac{2}{m}\left(E_0 + V_0\right)},$$

where b is the initial impact parameter, and b' is the impact parameter inside the sphere. Therefore,

$$b' = \sqrt{\frac{E_0}{E_0 + V_0}} b \equiv \eta b.$$

We label the scattering angle ϕ as shown, and define $\theta \equiv \pi - \phi$. If we extend the first and third rays, as shown in the diagram, we have an isosceles triangle with a vertex angle of θ. We define $\bar{b}$ to be the altitude of this triangle. From the geometry shown in Figure 8.16 we have

$$(b' + \bar{b}) \sin \frac{\theta}{2} = b,$$

or

$$\bar{b} = b\left(\csc\frac{\theta}{2} - \eta\right).$$

The longest side of the isosceles triangle is $2\sqrt{a^2 - b'^2}$. Therefore,

$$\tan\frac{\theta}{2} = \frac{\sqrt{a^2 - b'^2}}{\bar{b}} = \frac{\sqrt{a^2 - \eta^2 b^2}}{b\left(\csc\frac{\theta}{2} - \eta\right)}.$$

We can now solve for b in terms of θ, giving

$$b = \frac{a \sin\frac{\phi}{2}}{\sqrt{\eta^2 - 2\eta\cos\frac{\phi}{2} + 1}},$$

where we have substituted $\phi = \pi - \theta$. The differential cross section can then be obtained from the formula

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\phi}\left|\frac{db}{d\phi}\right| = \frac{a^2}{4\cos\frac{\phi}{2}} \frac{\left(\cos\frac{\phi}{2} - \eta\right)\left(1 - \eta\cos\frac{\phi}{2}\right)}{\left(\eta^2 - 2\eta\cos\frac{\phi}{2} + 1\right)^2}.$$

Example 8.13. For an inverse square force, the differential cross section only diverges when $\phi = 0$, which corresponds to particles with high impact parameter that are not deflected.

However, for other force laws, the differential cross section can diverge for non-zero angles. This indicates the presence of a "rainbow," where a large cross-sectional area of incoming particles is scattered to a narrow arc on a distant sphere. Determine the angle of the rainbow for a force given by the Lennard-Jones potential,

$$V(r) = 4\epsilon\left(\frac{\sigma^{12}}{r^{12}} - \frac{\sigma^6}{r^6}\right).$$

Solution: In Problem 14, you will show that the scattering angle for a force $F = \frac{k}{r^n}$ is

$$\phi \approx \frac{k}{mv_0^2 b^{n-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1}\theta\, d\theta.$$

Using the Lennard-Jones force

$$\boldsymbol{F}(r)\cdot\hat{\boldsymbol{r}} = -\frac{dV(r)}{dr} = -\frac{24\epsilon\sigma^6}{r^7} + \frac{48\epsilon\sigma^{12}}{r^{13}}$$

and the value for the integral given in Problem 14, we have a scattering angle

$$\phi \approx \frac{\pi\epsilon}{mv_0^2}\left(\frac{693\sigma^{12}}{64b^{12}} - \frac{15\sigma^6}{2b^6}\right).$$

The differential cross section is proportional to $\frac{db}{d\phi}$, so it will diverge whenever $\frac{d\phi}{db} = 0$. You can verify that this occurs when

$$b = \sigma\sqrt[6]{\frac{231}{80}}.$$

Evaluating ϕ for this value of b, we should have a rainbow angle of

$$\phi_r = -\frac{50\pi}{77}\frac{\epsilon}{E},$$

where $E = \frac{1}{2}mv_0^2$. Remember that our approximation is only valid when the scattering angle is small, so we must have $E \gg \epsilon$ in order for this result to hold. The agreement of this result with numerical simulations is shown in Figure 8.17.

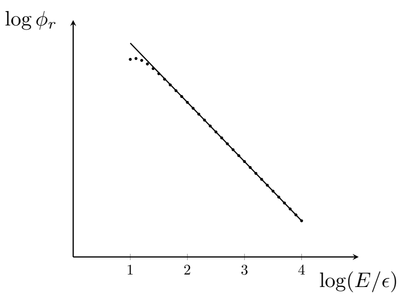


Figure 8.17: The simulation data (shown as points) agree with the approximate result when $E \gg \epsilon$.

Example 8.14. Determine the differential cross section for the scattering of an object under the potential $U(r) = \frac{\beta}{r^2}$.

Solution: We previously deduced the form of unbound orbits under the potential $U(r) = -\frac{\beta}{r^2}$. Setting $\beta \to -\beta$, the orbits under the potential $U(r) = \frac{\beta}{r^2}$ will take the form

$$\frac{1}{r} = \sqrt{\frac{2mE}{L^2 + 2m\beta}}\cos\left(\theta\sqrt{1 + \frac{2m\beta}{L^2}}\right),$$

where we set $\theta_0 = 0$. Clearly, the asymptotes of the orbit will occur when $\cos\left(\theta\sqrt{1+\frac{2m\beta}{L^2}}\right) = 0$. Therefore, the scattering angle is

$$\phi = \pi - \frac{\pi}{\sqrt{1 + \frac{2m\beta}{L^2}}}.$$

Using $L = mv_0 b$, we can solve for $b(\phi)$,

$$b = \sqrt{\frac{2\beta}{mv_0^2}}\frac{\pi - \phi}{\sqrt{\phi(2\pi - \phi)}}.$$

Therefore,

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\phi}\left|\frac{db}{d\phi}\right| = \frac{\pi^2\beta}{E\sin\phi}\left(\frac{\pi - \phi}{\phi^2(2\pi - \phi)^2}\right),$$

where $E = \frac{1}{2}mv_0^2$.

8.6 Systems of Many Bodies

So far, while studying central forces between two bodies, we have made the assumption that the mass of one body is much larger than the mass of the other. Even though Newton's third law requires that both bodies experience the same force, the force exerted by the smaller body on the massive body is not strong enough to significantly perturb the massive body. Therefore, we were able to assume the massive body remained fixed and solve for the motion of the smaller body. However, if the masses are comparable, we will need to account for motion of both bodies. This problem can be significantly simplified by a change of coordinates.

Figure 8.18 shows the change of coordinates. We assign vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ to the positions of two masses under the influence of a central force. We also define two new vectors: the position of the center of mass $\boldsymbol{R}$, and the relative position of the two masses $\boldsymbol{r}$. We express these vectors in terms of the masses of the individual position vectors of the objects:

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}$$

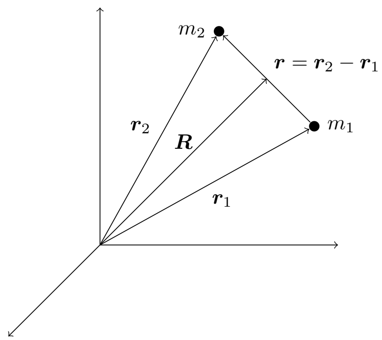


Figure 8.18: Two orbiting masses with an alternative set of position coordinates.

and

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1.$$

We can solve these equations for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, giving

$$\boldsymbol{r}_1 = \boldsymbol{R} - \left(\frac{m_2}{m_1+m_2}\right)\boldsymbol{r},$$

$$\boldsymbol{r}_2 = \boldsymbol{R} + \left(\frac{m_1}{m_1+m_2}\right)\boldsymbol{r}.$$

We will write the energy in terms of the new vectors. We know that

$$E = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1{}^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2{}^2 + U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|),$$

so we can substitute for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and obtain

$$E = \frac{1}{2}m_1\left(\dot{\boldsymbol{R}} - \left(\frac{m_2}{m_1+m_2}\right)\dot{\boldsymbol{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\boldsymbol{R}} + \left(\frac{m_1}{m_1+m_2}\right)\dot{\boldsymbol{r}}\right)^2 + U(r).$$

Simplifying, this becomes

$$E = \frac{1}{2}(m_1+m_2)\dot{\boldsymbol{R}}^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\dot{\boldsymbol{r}}^2 + U(|\boldsymbol{r}|).$$

The term $\frac{m_1m_2}{m_1+m_2}$ is called the reduced mass of the system, and is denoted μ. Note that it satisfies

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Our final result for the total energy is

$$E = \frac{1}{2}(m_1+m_2)\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 + U(r).$$

Since the system is not subject to any external force, the center of mass velocity $\dot{\boldsymbol{R}}$ is constant. Thus, the first term is constant and we can ignore it. This reduces the energy to a function of $\boldsymbol{r}$ alone, which we can treat as the energy of a one body system. This is the advantage of the change of coordinates we utilized.

In fact, we have done even more than reduce to a one-body problem: we have reduced to a problem which we have already solved. The energy is now equal to the energy of a planet of mass μ a distance r from the source of gravitation. Therefore, our solution to the Kepler problem is directly applicable to the two body problem. We need only perform the necessary changes of coordinates.

Example 8.15. A body with mass αM is stationary when an object with mass βM is launched with an initial velocity v_0 perpendicular to the line connecting the two masses. The distance between the masses is initially D. Determine the motion of both bodies and a constraint guaranteeing that their resulting orbit is unbound.

Solution: The total energy of the two body system is given by

$$E = \frac{1}{2}(m_1+m_2)\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 + U(r).$$

The center of mass velocity is given by $\frac{\beta}{\alpha+\beta}v_0$ and the reduced mass is $\frac{\alpha\beta}{\alpha+\beta}M$, so we can rewrite this as

$$E = \frac{M}{2}\frac{\beta^2}{\alpha+\beta}v_0^2 + \frac{M}{2}\frac{\alpha\beta}{\alpha+\beta}\left(\frac{dr}{dt}\right)^2 + U_e(r),$$

where $U_e(r) = \frac{L^2}{2\mu r^2} - \frac{C}{r} = \frac{L^2(\alpha+\beta)}{2M\alpha\beta r^2} - \frac{G\alpha\beta M^2}{r}$. If we let $E' = E - \frac{M}{2}\frac{\beta^2}{\alpha+\beta}v_0^2$, we see that we have an equation for a planetary orbit. The solution is

$$\begin{aligned} r &= \frac{L^2(\alpha+\beta)}{G\alpha^2\beta^2M^3}\frac{1}{1+\sqrt{1+\frac{2E'L^2(\alpha+\beta)}{G^2\alpha^3\beta^3M^5}}\cos\theta} \\ &= \frac{L^2(\alpha+\beta)}{G\alpha^2\beta^2M^3}\frac{1}{1+e'\cos\theta}. \end{aligned}$$

Now, we can rewrite our result in terms of D and v_0. We know that

$$L = \frac{\alpha\beta}{\alpha+\beta}Mv_0D$$

and

$$E' = \frac{M}{2}\beta v_0^2 - \frac{M}{2}\frac{\beta^2}{\alpha+\beta}v_0^2 - \frac{\alpha\beta M^2G}{D} = \frac{M}{2}\frac{\alpha\beta}{\alpha+\beta}v_0^2 - \frac{\alpha\beta M^2G}{D}.$$

Thus, the solution can be written as

$$r = \frac{v_0^2D^2}{GM(\alpha+\beta)}\frac{1}{1+e'\cos(\theta-\theta_0)}.$$

We solve for the phase shift θ_0 via the initial condition $r(\theta = 0) = D$, which gives us

$$\cos\theta_0 = \frac{1}{e'}\left(\frac{v_0^2 D}{GM(\alpha+\beta)} - 1\right).$$

The full equation of motion of the two masses can now be found by computing the location of the center of mass and adding the relative center of mass vector.

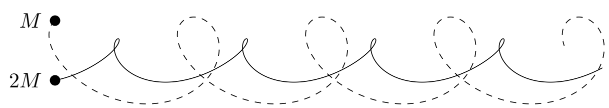
We still need to find a constraint such that the motion is unbound. This is possible if r diverges, which occurs when $E' \geq 0$. This implies that

$$\frac{M}{2}\frac{\alpha\beta}{\alpha+\beta}v_0^2 - \frac{\alpha\beta M^2 G}{D} \geq 0,$$

or

$$v_0^2 \geq \frac{2MG(\alpha+\beta)}{D}.$$

Figure 8.19 shows examples of possible orbits, corresponding to different values of E'.



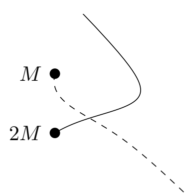


Figure 8.19: In the elliptical case (above), the two masses stay within some maximum distance of each other. In the hyperbolic case (below), the distance is unbound.

As you can see, even though a two-body problem is more difficult than a one-body problem, it is still solvable. However, carrying out this process for three or more bodies is generally not analytically tractable.

8.7 Binary Systems

In Chapter 4, we showed how an energy diagram can be used to gain a qualitative understanding of motion even when the equations of motion are not analytically tractable. This is one way of treating the three-body problem if the third mass is small enough that we can assume it does not affect the potential generated by the larger masses. In this section we will use this method

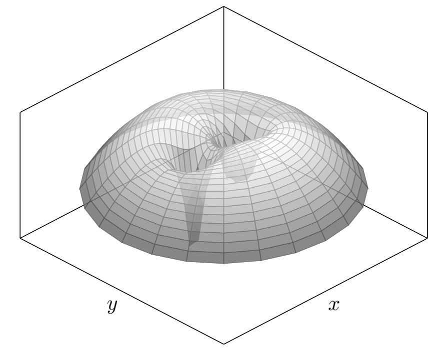


Figure 8.20: The potential energy per unit mass due to two massive bodies.

to understand the behavior of mass in a binary system. We will then look more closely at the influence of two gravitational bodies on each other.

Roche Lobes

Consider a binary system of two masses M_1 and M_2 separated by a distance r. We assume the system is bound, so

$$\frac{1}{2}\left(M_1 v_1^2 + M_2 v_2^2\right) - \frac{GM_1M_2}{r} < 0.$$

Then, as we showed in the previous section, the two bodies will orbit their center of mass. We will henceforth ignore this rotation, and consider only the static potential created by the two bodies. This amounts to working in a reference frame rotating with the bodies; you will complete such an analysis in detail in Problem 16, and the details of the dynamics will be developed in Chapter 12. For now, we simply recall from Example 8.6 that in the rotating frame, there is an outward centrifugal force with a potential per unit mass $U_c = -\frac{1}{2}\omega^2 r^2$. From our analysis of the two body problem, the angular velocity ω is given by

$$\omega^2 = \frac{G\mu}{(x_1 - x_2)^3},$$

where μ is the reduced mass.

The potential energy of the two bodies per unit test mass is the simple sum of their Newtonian potentials. We will place the origin at the center of mass and the bodies on the x axis, at positions x_1 and x_2 (where $M_1x_1 + M_2x_2 = 0$). Then the potential is given explicitly by

$$U(\boldsymbol{r}) = -\frac{GM_1}{\sqrt{(x-x_1)^2+y^2+z^2}} - \frac{GM_2}{\sqrt{(x-x_2)^2+y^2+z^2}} - \frac{1}{2}\omega^2 r^2.$$

This depends only on x and $y^2 + z^2$, so we will set $z = 0$ without loss of generality and work in the xy plane. A plot of the potential is shown in Figure 8.20.

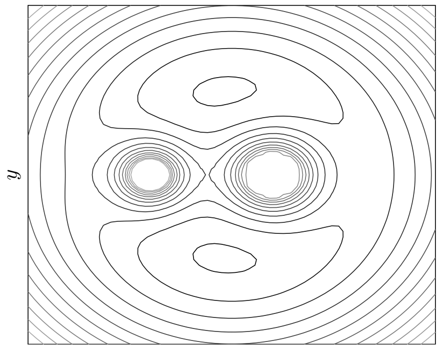


Figure 8.21: The equipotential lines of a binary system, including the teardrop shaped curves passing through the Lagrange point L_1.

This potential has several stationary points, known as *Lagrange points*. You will find the locations of all of the Lagrange points in Problem 16. For now, we will be focused on the Lagrange point L_1. This is the point on the line between the two masses where the gravitational forces balance the centrifugal force. Let $x_1 > 0$; then, the position x of the Lagrange point satisfies

$$\frac{GM_1}{(x-x_1)^2} - \frac{GM_2}{(x-x_2)^2} + \omega^2 x = 0.$$

Substituting for ω, we have

$$\frac{M_1}{(x-x_1)^2} - \frac{M_2}{(x-x_2)^2} + \frac{M_1 M_2 x}{(M_1+M_2)(x_1-x_2)^3} = 0.$$

Solving for x involves finding the zeroes of a quintic polynomial. We can see from the energy diagram that there will be three real solutions, with the middle solution corresponding to L_1.

From Figure 8.20, we see that there are roughly circular regions around each mass such that $U(\boldsymbol{r}) = U_{L1}$. Figure 8.21 shows these regions, along with other equipotential lines. They are known as the *Roche lobes*; an orbiting mass is gravitationally bound to an object in a binary system if it lies within the Roche lobe of the object.

The contour through the first Lagrange point is the critical transition between contours of lower energy, which are orbits bound to one mass, and contours of higher energy, which are orbits around the binary system as a whole. Note especially the contours of intermediate values of energy, which form horseshoe-like shapes. There is an asteroid known as Cruithne which follows this type of orbit in the Earth-Sun system; even though it is principally affected by the gravity of the Sun, it is sometimes referred to as Earth's second moon.

To make analytical progress, we will have to introduce an assumption which simplifies the quintic equation for the position of L_1. In Problem 16, you will assume $M_2 \ll M_1$. Here, we will look instead at the case where $M_1 = M_2 \equiv M$, and thus $|x_1| = |x_2| \equiv R$. In this case, L_1 clearly lies at the center of mass, as we can see by substituting $x = 0$ into the equation above. Moreover,

the orbital frequency of the binary system is $\omega^2 = \frac{GM}{16R^3}$. The potential of the Roche lobes is then

$$U(0) = -\frac{2GM}{R},$$

so the boundary of the lobes is at

$$\frac{R}{2\sqrt{(x-R)^2+y^2}} + \frac{R}{2\sqrt{(x+R)^2+y^2}} + \frac{r^2}{64R^2} = 1.$$

In polar coordinates, with $u = r/R$, this is

$$\left(1 - 2u\cos\theta + u^2\right)^{-1/2} + \left(1 + 2u\cos\theta + u^2\right)^{-1/2} + \frac{u^2}{32} = 2.$$

We can determine some features of the Roche lobe from this equation. For example, consider the limit $u \to 0$. Using a binomial expansion to second order in u, we find

$$2 - \frac{31}{32}u^2 + 3u^2\cos^2\theta = 2.$$

This determines the opening angle of the teardrop shape to be

$$\phi = 2\theta = 2\cos^{-1}\frac{\sqrt{31/6}}{4}.$$

Additionally, by setting $\theta = 0$ and solving for u, we find that the horizontal length of a Roche lobe is

$$L \approx 1.65R.$$

Since the outer layers of stars are fluid, the surface of a star will take the shape of an equipotential. Therefore, if a star is in a binary with another star of the same mass, it can expand until it takes the shape of the Roche lobe. If the outer layers expand beyond the Roche lobe, the mass will fall through the point L_1 into the Roche lobe of the other star. Of course, this does not account for the fact that the change in shape of the star will change the gravitational potential, which in turn changes the equipotential curves. Thus, we have completed only a first approximation; performing the calculation self-consistently would be far more complicated.

Tidal Forces

So far, we have mostly analyzed the behavior of point masses under gravitational forces. However, just like the objects causing strong gravitational forces, objects under the influence of gravitational forces are often large. This means that they will experience different forces at different points, which results in internal stress.

The forces resulting from the differential action of gravity at different points of a body are called *tidal forces*. The name refers to the most commonly observed effect, tides on Earth. The gravity of the moon has a different strength depending on the location on Earth, and the oceans change shape in response to this force.

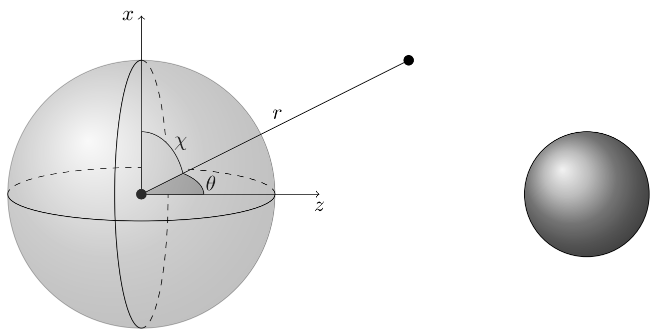


Figure 8.22: To compute the adjustment of Earth's gravitational field due to the moon, we use spherical coordinates centered on Earth and directed towards the moon.

To quantify this effect, let the masses of the moon and Earth be M_L and M_E respectively, let the distance from the center of Earth to the center of the moon be R, and let the radius of the Earth be r_E. We will work in a reference frame rotating about the center of mass of the system at the orbital frequency, so both Earth and the moon are stationary. The orbital frequency is

$$\omega = \sqrt{\frac{G(M_E + M_L)}{R^3}},$$

so there is a centrifugal potential per unit mass

$$U_c = -\frac{1}{2}\omega^2 s^2,$$

where s is measured from the axis of rotation. In addition we must add the gravitational potentials of Earth and the moon. We will use spherical coordinates centered on Earth, as shown in Figure 8.22. Then the total gravitational potential is given by

$$U_g = -\frac{GM_E}{r} - \frac{GM_L}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}.$$

To transform the centrifugal potential into these coordinates, let x be the distance of the center of mass from the center of Earth. Then we have

$$s^2 = (x - r\cos\theta)^2 + (r\sin\theta\sin\phi)^2 = x^2 - 2xr\cos\theta + r^2(1 - \sin^2\theta\cos^2\phi).$$

We will define $\cos\chi = \sin\theta\cos\phi$; geometrically, χ is the angle between $\boldsymbol{r}$ and the axis of Earth parallel to the rotation axis. In total, the potential per unit mass is

$$\begin{aligned} U(\boldsymbol{r}) = &-\frac{GM_E}{r} - \frac{GM_L}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} \\ &- \frac{G(M_E + M_L)}{2R^3}\left(x^2 - 2xr\cos\theta + r^2\sin^2\chi\right). \end{aligned}$$

We will now approximate the lunar potential using the method of Section 8.3. Keeping up to quadrupole terms, we have

$$-\frac{GM_L}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} \approx -\frac{GM_L}{R}\left(1 + \frac{r}{R}\cos\theta + \frac{r^2}{2R^2}\left(3\cos^2\theta - 1\right)\right).$$

Therefore,

$$\begin{aligned} U(\boldsymbol{r}) = &-\frac{GM_E}{r} - \frac{GM_L}{R} - \frac{G(M_E + M_L)}{2R^3}(x^2 + r^2\sin^2\chi) \\ &- \frac{Gr}{R^2}\left(M_L - \frac{(M_E + M_L)x}{R}\right)\cos\theta - \frac{GM_L r^2}{2R^3}\left(3\cos^2\theta - 1\right). \end{aligned}$$

Since x is the position of the center of mass, we have $x = \frac{M_L}{M_E + M_L}R$. Upon substituting this, the $\cos\theta$ term vanishes. Letting U_0 denote the constant terms, we are left with

$$U(\boldsymbol{r}) = U_0 - \frac{GM_E}{r} - \frac{G(M_E + M_L)}{2R^3}(r\sin\chi)^2 - \frac{GM_L r^2}{2R^3}\left(3\cos^2\theta - 1\right).$$

We recognize the third term as the centrifugal potential corresponding to the spin of Earth in the rotating frame. The fourth term is the correction due to tidal forces.

To determine the adjustment to the equipotential surface, we have to determine the value of U at the surface. We can do this by enforcing that the volume of Earth remains constant – therefore, if the equator bulges outwards, the poles must compress inward. This implies

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{r_E(1+q)} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{4}{3}\pi r_E^3,$$

where $q = \frac{r}{r_E} - 1$. If we expand the potential to first order in q, we have

$$U(\boldsymbol{r}) = U_0 - \frac{GM_E}{r_E}(1 - q) - \frac{G(M_E + M_L)r_E^2}{2R^3}\left(\sin^2\chi + \alpha(3\cos^2\theta - 1)\right)(1 + 2q),$$

where $\alpha = \frac{M_L}{M_E + M_L}$. Solving for q, we obtain

$$q = \frac{\tilde{U} + \frac{G(M_E + M_L)r_E^2}{2R^3}\left(\sin^2\chi + \alpha(3\cos^2\theta - 1)\right)}{\frac{GM_E}{r_E} - \frac{G(M_E + M_L)r_E^2}{R^3}\left(\sin^2\chi + \alpha(3\cos^2\theta - 1)\right)},$$

where $\tilde{U}$ is an unknown constant that specifies the potential at Earth's surface. Since Earth's radius is much smaller than the Earth-moon distance, the second term in the denominator can be neglected, and we have

$$q = \frac{r_E}{GM_E}\left(\tilde{U} + \frac{G(M_E + M_L)r_E^2}{2R^3}\left(\sin^2\chi + \alpha(3\cos^2\theta - 1)\right)\right).$$

If we simplify the integral to calculate the total volume, we have

$$\frac{r_E^2}{3}\int_0^{2\pi}\int_0^{\pi}(1 + q)^3 \sin\theta \, dr \, d\theta \, d\phi = \frac{4}{3}\pi r_E^3.$$

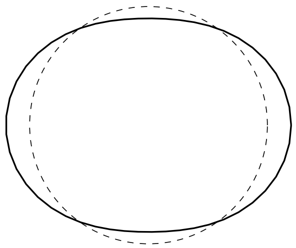


Figure 8.23: The moon's gravitational pull produces tidal bulges and polar depressions in the Earth's ocean, shown greatly exaggerated here.

Using $(1+q)^3 \approx 1+3q$, we see that

$$\int_0^{2\pi}\int_0^{\pi} q(\theta,\phi)\sin\theta\, dr\, d\theta\, d\phi = 0.$$

Integrating with the explicit form for q, we obtain

$$4\pi\left(\tilde{U} + \frac{2}{3}\frac{G(M_E+M_L)r_E^2}{2R^3}\right) = 0.$$

This allows us to determine the unknown constant, and therefore the general form for the equipotential surface,

$$q = \frac{(M_E+M_L)r_E^3}{2M_E R^3}\left(\sin^2\chi + \alpha(3\cos^2\theta - 1) - \frac{2}{3}\right).$$

The resulting deformation of Earth is shown in Figure 8.23. The bulge closest to the moon is simple to explain: it results from the pull of the moon's gravity on the water. The other bulge is a result of the relative weakness of the moon's gravity on the far side of Earth. Since tidal forces result from differences in gravity, the strength of the tidal force of an object of mass M at distance R on an object of mass m and size r scales as

$$|\boldsymbol{F}_t| \approx r|\boldsymbol{\nabla F}_g| = \frac{2GMmr}{R^3}.$$

From this, we can compare the tidal forces of the sun to those of the moon. The solar mass is approximately 3×10^7 times the lunar mass, but the moon is about 400 times closer than the sun. Therefore, the tidal forces of the sun are about half the strength of those of the moon on Earth. When the sun and moon align, the combined tidal force leads to stronger tides, called spring tides. When the sun and moon are on opposite sides of Earth, their tidal forces counteract, resulting in weaker tides, called neap tides.

Stability of Satellites

Although the tidal forces exerted by the moon on Earth are the most easily visible, Earth also exerts significant tidal forces on the moon. We can see from the above formula that if the moon

were to orbit closer to Earth, it would experience even stronger tidal forces. However, its own gravity would not change. Therefore, there must be a distance at which the tidal forces of Earth become strong enough to pull the moon apart.

We can roughly compute the radius at which this occurs with a simple calculation. Consider a mass Δm on the surface of the moon. It is bound to the moon by the Newtonian force of gravity, and it is pulled away by the tidal force we derived above. Therefore, the minimum radius at which it can orbit is given by

$$\frac{GM_L\Delta m}{r_L^2} = \frac{2GM_E\Delta m\, r_L}{R^3}.$$

Solving for R, we have

$$R = \left(\frac{2M_E r_L^3}{M_L}\right)^{1/3} = \sqrt[3]{2}r_E\left(\frac{\rho_E}{\rho_M}\right)^{1/3}.$$

This distance is called the *Roche limit*, and it exists for any system of orbiting bodies. When a satellite held together by gravity crosses within the Roche limit, it becomes unstable and falls apart. Planetary rings are often formed in this way.

Example 8.16. A planet of mass m and radius r orbits a star of mass M in a circle of radius R. The planet is a sphere with a density proportional to $1+\epsilon\frac{z}{r}$, where z is the distance along the planet's symmetry axis as measured from its center of mass, so one side of the planet is more massive than the other. Assume that the planet has zero angular velocity at $t=0$. Show that the planet becomes "tidally locked", with the more massive side always facing the star. Then find the Roche limit for a tidally locked planet.

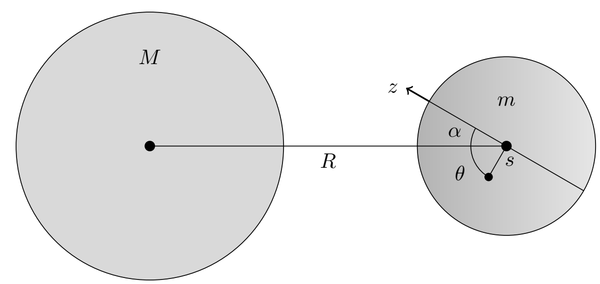


Figure 8.24: A planet with a non-uniform density becomes tidally locked with a star.

Solution: The gravitational potential of the planet due to its own field will be the same for any orientation of the planet, so we will ignore it. We will use spherical coordinates shown in Figure 8.24, where θ measures the polar angle from the star-planet axis, and α is the polar angle of the axis of symmetry of the density profile. The potential energy of the planet is then given by

$$U(\alpha) = -\iiint \frac{GM}{\sqrt{R^2+s^2-2Rs\cos\theta}}\rho_0\left(1+\epsilon\frac{s\cos\zeta}{R}\right) dV,$$

where ζ measures the angle to the symmetry axis of the planet. Letting ϕ be the azimuthal angle of the spherical system, it is simple to show that

$$\cos\zeta = \sin\theta\cos\phi\sin\alpha + \cos\theta\cos\alpha.$$

Expanding the potential up to terms of order $\frac{s^2}{R^2}$, we find

$$U(\alpha) = -\frac{GM\rho_0}{R}\iiint\left(1+\frac{s}{R}(\cos\theta+\epsilon\cos\zeta)+\frac{s^2}{2R^2}(3\cos^2\theta-1)+\frac{s^2}{R^2}\epsilon\cos\theta\cos\zeta\right)dV.$$

After integration, the constant term gives the volume which adds only a constant to the potential, and the second term vanishes. The quadrupole term similarly gives a constant factor, so the interesting term is the last one. Ignoring the constant $\frac{GMm}{R}$ term, we have

$$U(\alpha) = -\epsilon\frac{GM\rho_0 r^5}{5R^3}\int_0^{2\pi}\int_0^{\pi}\sin\theta\cos\theta(\sin\theta\cos\phi\sin\alpha+\cos\theta\cos\alpha)\,d\theta\,d\phi.$$

The first term vanishes after integration over ϕ. The second term is nonzero, and gives

$$U(\alpha) = -\epsilon\frac{GMmr^2}{5R^3}\cos\alpha.$$

As we would expect, the potential is minimized when $\alpha = 0$ and the massive side of the planet is facing the star. The torque on the planet is

$$\tau(\alpha) = -\frac{dU}{d\alpha} = -\epsilon\frac{GMmr^2}{5R^3}\sin\alpha.$$

The planet orbits the star with an angular velocity

$$\Omega = \sqrt{\frac{G(M+m)}{R^3}}.$$

Let the spin angular velocity of the planet be $\omega(t)$. Then the relative orientation of the planet changes according to

$$\dot{\alpha} = \Omega - \omega(t).$$

We can also determine $\ddot{\alpha}$ using the torque. You can verify that the planet has the same moment of inertia as a uniform sphere of the same mass, so

$$\ddot{\alpha} = \frac{\tau(\alpha)}{\frac{2}{5}mr^2} = -\epsilon\frac{GM}{2R^3}\sin\alpha.$$

Differentiating the first equation and comparing with the second, we find

$$\dot{\omega}(t) = \epsilon\frac{GM}{2R^3}\sin\alpha.$$

Combining this with the equation for $\dot{\alpha}$, we have a coupled system of first-order differential equations. It is clear that equilibrium is reached when $\omega = \Omega$ and $\alpha = 0$ or $\alpha = \pi$, and $\alpha = 0$ is the stable equilibrium.

To find the Roche limit for the tidally locked planet, we account for the centrifugal force as well as the tidal force. A mass Δm on the planet is bound by the gravity of the planet, and pulled away by the tidal force as well as the centrifugal force. This gives

$$\frac{Gm\Delta m}{r^2} = \frac{2GM\Delta mr}{R^3} + \frac{\omega^2 r\Delta m}{2}.$$

Using $\omega^2 = \Omega^2 = \frac{2G(M+m)}{R^3}$, this is

$$\frac{m}{r^2} = \frac{(3M+m)r}{R^3}.$$

The Roche limit is thus

$$R = \left(\frac{(3M+m)r^3}{m}\right)^{1/3} \approx \sqrt[3]{3}r_S\left(\frac{\rho_S}{\rho_P}\right)^{1/3},$$

where r_S and ρ_S are the radius and density of the star and ρ_P is the average density of the planet.

Key Concepts

- Central forces conserve energy and angular momentum.
- Newton's shell theorem gives the gravitational field of large spherical bodies.
- Kepler's three laws state properties of orbits under gravitational forces.
- Scattering by a central force can be described in terms of a differential cross section.
- Two body central force problems can be solved using center of mass coordinates.
- Tidal forces due to the moon leads to the bulging of Earth's surface.

8.8 Problems

1. A particle is confined to move along the surface of a hemispherical cavity. Using the conservation of angular momentum, show that the particle behaves as if it is in the effective potential
$$U_e(\theta) = \frac{L^2}{2m\ell^2\sin^2\theta} - mg\ell\cos\theta,$$
where m is the mass of the particle, L is its angular momentum, ℓ is the radius of the cavity, and θ refers to the spherical coordinate. Determine the equilibrium position of the mass, as well as the frequency of small oscillations about this equilibrium. Indicate using a diagram how the mass evolves in time given some initial conditions.

2. The orbit of a mass is given by $r = a(1+\cos\theta)$.