To find the Roche limit for the tidally locked planet, we account for the centrifugal force as well as the tidal force. A mass Δm on the planet is bound by the gravity of the planet, and pulled away by the tidal force as well as the centrifugal force. This gives

$$\frac{Gm\Delta m}{r^2} = \frac{2GM\Delta mr}{R^3} + \frac{\omega^2 r\Delta m}{2}.$$

Using $\omega^2 = \Omega^2 = \frac{2G(M+m)}{R^3}$, this is

$$\frac{m}{r^2} = \frac{(3M+m)r}{R^3}.$$

The Roche limit is thus

$$R = \left(\frac{(3M+m)r^3}{m}\right)^{1/3} \approx \sqrt[3]{3}r_S\left(\frac{\rho_S}{\rho_P}\right)^{1/3},$$

where r_S and ρ_S are the radius and density of the star and ρ_P is the average density of the planet.

Key Concepts

- Central forces conserve energy and angular momentum.
- Newton's shell theorem gives the gravitational field of large spherical bodies.
- Kepler's three laws state properties of orbits under gravitational forces.
- Scattering by a central force can be described in terms of a differential cross section.
- Two body central force problems can be solved using center of mass coordinates.
- Tidal forces due to the moon leads to the bulging of Earth's surface.

8.8 Problems

1. A particle is confined to move along the surface of a hemispherical cavity. Using the conservation of angular momentum, show that the particle behaves as if it is in the effective potential
$$U_{\mathrm{e}}(\theta) = \frac{L^2}{2m\ell^2\sin^2\theta} - mg\ell\cos\theta,$$
where m is the mass of the particle, L is its angular momentum, ℓ is the radius of the cavity, and θ refers to the spherical coordinate. Determine the equilibrium position of the mass, as well as the frequency of small oscillations about this equilibrium. Indicate using a diagram how the mass evolves in time given some initial conditions.

2. The orbit of a mass is given by $r = a(1+\cos\theta)$.

(a) Determine the central force which leads to this orbit.

(b) Determine the total cross section of capture if the object is under the influence of this force and starts out with a speed v at infinity. You do not need to find the differential cross section to do this part.

3. Consider a mass orbiting in a circular trajectory under the influence of an attractive central force $F(r) = -\frac{k}{r^n}$. What constraint must be placed on n for the orbits to be stable?

4. Consider the Earth-sun system. To account for interstellar dust, we assume that the universe is isotropically filled with fine dust of density ρ. We will also assume that the mass of the earth is m and the mass of the sun is M.

(a) Show that the net central force acting on the earth is

$$F_{net} = -\frac{GMm}{r^2} - mkr,$$

where k is a constant for you to find. Assume that Earth does not lose energy or pick up dust as it orbits the sun.

(b) Show that for small radial perturbations from a stable circular orbit of radius R_0, the motion of Earth can be modeled as a precessing ellipse. Assuming that the force due to the dust is much smaller than the force due to the sun, show that the frequency at which the ellipse precesses is given by

$$\Omega \approx 2\pi\rho\sqrt{\frac{R_0^3 G}{M}}.$$

5. When one adds a correction δU to the potential $U = -\frac{\alpha}{r}$, the bound orbits of a planet-star system are no longer closed. Instead, as you saw in the previous problem, the orbits precess.

(a) Show that we can write the angle as a function of radius in the following way:

$$\Delta\theta = -2\frac{\partial}{\partial L}\int_{r_{min}}^{r_{max}} \sqrt{2m(E-U) - \frac{L^2}{r^2}}\, dr.$$

This allows us to expand the integrand as a Taylor series to first order in δU. Note that writing $\Delta\theta$ in terms of a derivative helps to avoid divergences in the integral.

(b) Show that after a full period the ellipse rotates by a small angle, which is approximately given by

$$\delta\theta \approx \frac{\partial}{\partial L}\left(\frac{2m}{L}\int_0^{\pi} r^2 \delta U \, d\theta\right)$$

(c) Calculate this angle for $\delta U = \frac{\beta}{r^2}$ and $\delta U = \frac{\gamma}{r^3}$.

6. An object elliptically orbits a planet under the influence of gravity. At the perigee (the closest point to the planet), the object fires its thrusters with an impulse I in the radial direction, transitioning it to another elliptical orbit. Determine the new eccentricity, semi-major axis, and orientation of the orbit in terms of the previous values, e and a. You may also express your answers in terms of the initial energy of the object E, as well as the initial angular momentum about the planet L.

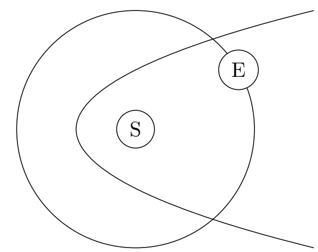

Figure 8.25: A comet spends some time inside Earth's orbit as it passes the sun.

7. Figure 8.25 depicts a comet moving in a parabolic trajectory about the sun, reaching a minimum distance before flying off to infinity. Earth orbits around the sun, with an orbital period T. What is the maximum time the comet can spend within the circular trajectory defining Earth's orbit? Assume that the interaction between the comet and Earth is negligible.

8. Systems interacting under an attractive central potential obey the *virial theorem*. This problem will explore the virial theorem and one of its interesting applications.

(a) Two masses are bound by a harmonic potential $U = \frac{1}{2}kr^2$, where r is the distance between them. Show that in the center of mass frame $\langle K\rangle = \langle U\rangle$, where K is the kinetic energy and $\langle\cdot\rangle$ denotes a time average.

(b) Determine the conditions under which the time average of a time derivative must converge to zero, i.e., $\langle \frac{dA}{dt}\rangle = 0$ for some variable A.

(c) Compute the moment of inertia of an N-body system with interaction potential $U(r) = kr^n$, and use the result of part (b) to show that $\langle K\rangle = \frac{n}{2}\langle U\rangle$.

(d) Determine the mass of a gravitating cluster of galaxies, assuming its mass is distributed approximately uniformly over a sphere of radius R and that the average squared velocity of the galaxies is v^2.[iv]

9. Using the conservation of angular momentum, rewrite the equation of motion for the relative position in a two-body orbit as

$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = \frac{\mu r^4}{L^2}F(r) + r.$$

10. N particles are placed in an N-gon whose radius is R. The particles all have equal mass m.

(a) Show that, if the masses all interact through a gravitational force, they can be modeled as an effective mass in a circular orbit about the center of the N-gon. Determine this effective mass and the frequency of the orbit.

[iv] This method was used by Fritz Zwicky to determine the mass of the Coma cluster. He found that its ratio of mass to luminosity is on the order of 500 times that of the Sun. Given that the Sun is typical of luminous matter in the universe, this was the first historical evidence for dark matter.

(b) Assume the masses are all stopped and released from rest. Determine the amount of time it takes for the masses to collide. Take the limit as N goes to infinity and explain your answer. Assume that the total mass remains finite in this limiting process.

11. The Laplace-Runge-Lenz vector for a Kepler orbit is defined to be

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - mC\hat{\boldsymbol{r}}.$$

(a) Show that $\boldsymbol{A}$ is constant for two bodies interacting with gravity.

(b) Show that $e = \frac{|\boldsymbol{A}|}{C}$.

12. A satellite is launched from the North pole and lands at the equator. If the satellite is launched with the minimum possible velocity, what is the angle it is launched at? What is this minimum velocity?

13. The gravitational potential per mass can be expanded in a Taylor series as

$$u(\boldsymbol{r}_0 + \boldsymbol{r}) = U(\boldsymbol{r}_0) + \sum_{i=1}^{3} \left(\frac{\partial U}{\partial r_i}\right)_{\boldsymbol{r}=\boldsymbol{r}_0} r_i + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial^2 U}{\partial r_i \partial r_j}\right)_{\boldsymbol{r}=\boldsymbol{r}_0} r_i r_j + \dots.$$

Show that, if $u(\boldsymbol{r}) = \frac{1}{|\boldsymbol{r}|}$, then the second partial derivatives are proportional to the components of the quadrupole moment matrix. Use this to verify the following expression for the gravitational quadrupole correction:

$$-\frac{G}{2r^3}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij} n_i n_j.$$

14. The scattering angle can be determined exactly for inverse square forces, but this is not possible in general. Find an approximation to the scattering angle due to a force $F = \frac{k}{r^n}$ under the assumption that the impact parameter is large and so there is little deflection. Use the integral

$$\int_0^{\frac{\pi}{2}} \cos^k\theta \, d\theta = \begin{cases} \frac{\left(\frac{k-1}{2}\right)! 2^{\frac{k-1}{2}}}{k\cdot(k-2)\cdot(k-4)\cdots(3)(1)} & \text{if k odd} \\ \frac{\pi(k-1)\cdot(k-3)\cdots(3)(1)}{\left(\frac{k}{2}\right)! 2^{\frac{k+2}{2}}} & \text{if k even} \end{cases}$$

15. Consider a potential of the form

$$V(r) = \begin{cases} 0 & r > a \\ \frac{k}{r} - \frac{k}{a} & r \le a \end{cases},$$

which is a truncated form of a repulsive gravitational potential. Determine the impact parameter as a function of the scattering angle, assuming that the initial energy of the particle is E. Find the value of the impact parameter that separates forward and backward scattering.

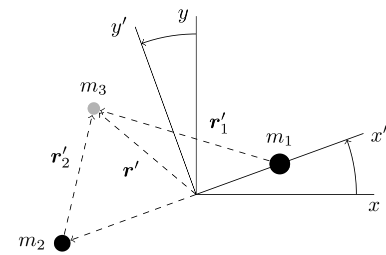

Figure 8.26: A coordinate system for analyzing the restricted three-body problem.

16. This problem involves a restricted case of the three body problem which is analytically solvable. Two masses are fixed to rotate in a circular orbit about their center of mass, while a third small mass is influenced by the former two. We will use the coordinate system shown in Figure 8.26 and take the distance between m_1 and m_2 to be the unit length. We will also assume that $G(m_1 + m_2) = 1$ in appropriate units, and define $\alpha \equiv \frac{m_2}{m_1+m_2}$.

(a) In the rotating reference frame for which m_1 and m_2 are stationary and m_1 is at the origin, determine a modified version of Newton's laws for the small mass m_3.

(b) Your equations will depend on the velocity of m_3 because of the rotating reference frame. Show that, neglecting the velocity terms, the equations of motion will describe a mass in an effective potential

$$U_{\text{eff}}(x, y) = -\frac{1-\alpha}{\sqrt{(x-\alpha)^2 + y^2}} - \frac{\alpha}{\sqrt{(x+1-\alpha)^2 + y^2}} - \frac{x^2+y^2}{2}.$$

(c) By taking the gradient of this potential and setting the result to zero, show that we get approximately five solutions for $\alpha \ll 1$, three of which are saddle points and two of which are maxima. These are known as the *Lagrange points*:

$$L_1 = \left(-1 + \sqrt[3]{\frac{\alpha}{3}}, 0\right), \quad L_2 = \left(-1 - \sqrt[3]{\frac{\alpha}{3}}, 0\right), \quad L_3 = \left(1 + \frac{5\alpha}{12}, 0\right),$$
$$L_4 = \left(\frac{1}{2} - \alpha, \frac{\sqrt{3}}{2}\right), \quad L_5 = \left(\frac{1}{2} - \alpha, -\frac{\sqrt{3}}{2}\right).$$

The apparent instabilities at these points are partially prevented by the velocity-dependent terms which we dropped to determine the effective potential.

(d) Show that the quantity

$$C = -2U_{\text{eff}}(x, y) - \dot{x}^2 - \dot{y}^2,$$

is conserved throughout the evolution of m_3. This is known as the Jacobi integral, and its conservation gives rise to a set of first order differential equations that can be used to plot the motion of m_3. Numerically solving this set of first order differential equations is computationally less taxing than solving Newton's laws.

17. A rocket orbits in a circle around a planet of mass M at a radius r_1. It needs to reach an orbit with radius $r_2 > r_1$, while minimizing the total change in velocity (and thus minimizing the total fuel required for the transfer). There are two possible strategies for this orbit transfer, outlined below and in Figure 8.27:

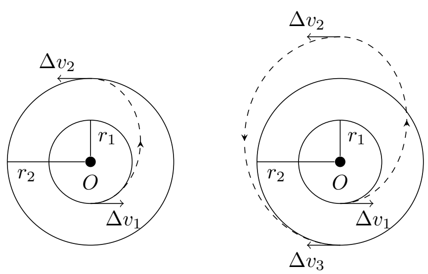

Figure 8.27: The Hohmann transfer (left) and the bi-elliptic transfer (right).

(a) The Hohmann transfer: transfer the object to an elliptic orbit with a velocity change Δv_1. At the apogee of the elliptic orbit, transfer the object to the desired circular object with another velocity change Δv_2.

(b) The bi-elliptic transfer: transfer the object to an elliptic orbit with a velocity change Δv_1. At the apogee of the elliptic orbit, transfer the object to another elliptic orbit with a velocity change Δv_2. At the apogee of this elliptic orbit, transfer the object to the desired circular orbit through a final velocity change Δv_3. Let $r_a > r_2$ be the distance from the planet to the apogee of both elliptic orbits.

Determine the total change in velocity for both transfer strategies. Show that the first strategy is more effective unless $\frac{r_2}{r_1} \geq 11.94$. (Hint: to show this, select r_a such that the total change in velocity for the bi-elliptic strategy is a minimum.)